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Degeneracy of Resonances: Branch Point and Branch Cuts in Parameter Space

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The rich phenomenology of crossings and anticrossings of energies and widths, observed in an isolated doublet of resonances when one control parameter is varied, is fully explained in terms of the topological properties of the energy hypersurfaces close to the degeneracy point. The hypersurface representing the complex resonance eigenvalues, as functions of the control parameters, has an algebraic branch point of rank one, and branch cuts in its real and imaginary parts, in parameter space. Associated with this singularity in parameter space, the scattering matrix, $S_\ell(E)$, and the Green's function, $G_\ell^{(+)}(k; r, r')$, have one double pole in the unphysical sheet of the complex energy plane. We characterize the universal unfolding or deformation of any degeneracy point of two unbound states in parameter space by means of a universal 2-parameter family of functions which is contact equivalent to the pole position function of the isolated doublet of resonances at the exceptional point and includes all small perturbations of the degeneracy condition up to contact equivalence.

KEY WORDS: multiple resonances degeneracies; non-relativistic scattering theory phases: topological; Berry phase

1. INTRODUCTION

Two level mixing of coherent energy eigenstates of a quantum system is a well known and important phenomenon (Cohen-Tannoudji *et al.*, 1973; Feynmann *et al.*, 1970; Landau and Lifshitz, 1974). In this paper, we will be concerned with the mixing and degeneracy of the two energy eigenstates in an isolated doublet of unbound states of a quantum system depending on two control parameters.

In the case of bound states of a Hermitian Hamiltonian depending on parameters, the energy eigenvalues are real and the two level mixing leads to the familiar phenomenon of energy level repulsion and avoided level crossings when a single parameter is varied. J. von Neumann and E. P. Wigner, in their celebrated theorem

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(von Neumann and Wigner, 1929), explained that, in the absence of symmetry, true crossing or degeneracy requires the variation of a number of parameters equal to the codimension of the degeneracy which, in the general case, is three. Later, E. Teller (1937), showed that "if the parameters are X, Y and Z, the two degenerating levels correspond to the two sheets of an elliptic double cone in the (X, Y, Z, E) space, near the degeneracy," this constitutes the diabolic crossing scenario (Berry, 1985), of the levels $\mathcal{E}_{\pm}$. For a recent review on diabolical conical intersections, see D. R. Yarkoni (1996).

In the case of unbound states, the energy eigenvalues are complex, this fact opens a rich variety of possibilities, namely, crossings or anticrossings of energies and widths.

In particular, a joint crossing of energies and widths produces a true degeneracy of resonance energy eigenvalues in a physical system depending on only two real parameters (Mondragón and Hernández, 1993).

Recently, a great deal of attention has been given to the rich scenario of crossings and anticrossings of energies and widths of resonance energy eigenvalues in the case of unbound states of a quantum system. Novel effects have been found which attracted considerable theoretical (Kylstra and Joachain, 1998; Friedrich and Wintgen, 1985; Hernández and Mondragón, 1994) and recently, also experimental interest (von Brentano, 1990, 1996).

The crossing and anticrossing of energies (frequencies) and widths of two interacting resonances in a microwave cavity were carefully measured by P. von Brentano, who also discussed the generalization of the von Neumann Wigner theorem from bound to unbound states (von Brentano and Philipp, 1999; Philipp *et al.*, 2000; von Brentano, 2002). The problem of the characterization of the singularities of the energy surfaces at a degeneracy of resonances arises naturally in connection with the topological phase of unbound states which was predicted by Hernández, Jáuregui, and Mondragón (1992); Mondragón and Hernández (1996, 1998), and, later and independently, by W. D. Heiss (1999), and which was recently measured by the Darmstadt group (Dembowski *et al.*, 2001, 2003).

A number of examples of double poles in the scattering matrix of simple quantum mechanical systems have been recently described. The formation of resonance double poles of the scattering matrix in a two channel model with square well potentials was described by Vanroose *et al.* (1997). Hernández *et al.* (2000) investigated a one channel model with a double $\delta-$barrier potential and showed that a double pole of the S-matrix can be induced by tuning the parameters of the model. Generalizations of the double barrier potential model to the case of finite width barriers were proposed and discussed by W. Vanroose (2001) and Hernández *et al.* (2003a).

Korsch and Mossman (2003) made a detailed investigation of degeneracies of resonances in a symmetric double $\delta-$well in a constat Stark field. Keck *et al.* (2003) extended and generalized the discussion of the Berry phase of resonance states, from the case of unbound states of a hermitian Hamiltonian given in Hernández *et al.* (1992); Mondragón and Hernández (1996, 1998), to the case of unbound states of non-hermitian Hamiltonians.

The general theory of Gamow or resonant eigenfunctions associated with multiple poles of the scattering matrix and Jordan blocks in the complex energy representation of the resolvent operator was developed by I. Antoniou (1998), A. Bohm (1997) and Hernández *et al.* (2003b).

In this paper, we will discuss the mixing and degeneracy of an isolated doublet of unbound states in the framework of the theory of the analytical properties of the radial wave functions. A generalization of the von Neumann-Wigner theorem and the Teller geometric construction from bound to unbound states will also be given.

The outline of our paper is as follows: the pole position function is introduced and its analytical properties are discussed in terms of a simple and explicit but very accurate contact equivalent approximant in Section 2. The universal unfolding of the Jost function at the degeneracy of resonances and a family of functions which is a universal unfolding of the degeneracy and is also contact equivalent to the exact resonance energy eigenvalues are given in Section 3. Section 4 is devoted to a discussion of crossings and anticrossings of energies and widths of the resonances in the mixings of an isolated doublet of unbound states. The trajectories of the S-matrix poles in the complex energy plane are derived and the changes of identity of the resonant poles in the vicinity of a crossing of unbound states are discussed in Section 5. We end our paper in Section 6 with a summary of results and some conclusions.

2. BRANCH POINTS AND BRANCH CUTS OF DEGENERATE ENERGY EIGENVALUES IN PARAMETER SPACE

In this section we will consider the resonance energy eigenvalues of a radial Schrödinger Hamiltonian, $H_r^{(\ell)}$, with a potential $V(r; x_1, x_2)$ which is a short range function of the radial distance, r, and depends on at least two external control parameters (x_1, x_2). The resonance energy eigenvalues are branches of a multivalued function of the external parameters (Newton, 1982). When the potential $V(r; x_1, x_2)$ is short ranged and has two regions of trapping, the physical system may have isolated doublets of resonances which may become degenerate for some special values of the control parameters. For example, a double square barrier potential has isolated doublets of resonances which may become degenerate for some special values of the heights and widths of the barriers (Hernández *et al.*, 2000; Vanroose, 2001; Hernández *et al.*, 2003a). In this case, the corresponding two energy eigenvalues, say $\mathcal{E}_n(x_1, x_2)$ and $\mathcal{E}_{n+1}(x_1, x_2)$, are equal (cross or coincide) for those special values of the control parameters.

It will be shown that, when the physical system has an isolated doublet of resonances which become degenerate for some exceptional values of the external parameters, (x_1^*, x_2^*), the energy hypersurfaces representing the complex resonance energy eigenvalues as functions of the control parameters have an algebraic branch point of square root type (rank one) in parameter space. The analytical structure of the singularity of the energy eigenvalues as functions of the real control parameters will be worked out and discussed in detail.

The regular and physical solutions of the radial Schrödinger equation with the Hamiltonian $H_r^{(\ell)}(x_1, x_2)$ are functions of the radial distance r, the wave number k and the control parameters (x_1, x_2). When necessary, we will stress this last functional dependence by adding the control parameters (x_1, x_2) to the other arguments after a semicolon.

The energy eigenvalues $\mathcal{E}_n$ of the Hamiltonian $H_r^{(\ell)}$ are obtained from the zeroes of the Jost function, $f(-k; x_1, x_2)$,

$$\mathcal{E}_n = \frac{\hbar^2}{2m} k_n^2, \tag{1}$$

where k_n is such that

$$f(-k_n; x_1, x_2) = 0. \tag{2}$$

When k_n lies in the fourth quadrant of the complex k-plane,

$$\mathrm{Re}\, k_n > 0 \quad \text{and} \quad \mathrm{Im}\, k_n < 0, \tag{3}$$

the corresponding energy eigenvalue $\mathcal{E}_n$, is a complex resonance energy eigenvalue.

The condition (2) defines, implicitly, the functions $k_n(x_1, x_2)$ as branches of a multivalued function (Newton, 1982) which will be called the pole position function.

In the case of a set $\{k_n(x_1, x_2)\}$ of isolated simple zeroes of the Jost function, Eq. (2) may, in principle, be solved for each branch $k_n(x_1, x_2)$ without ambiguity,

$$k_n(x_1, x_2) = f^{-1}(0; x_1, x_2). \tag{4}$$

Each branch $k_n(x_1, x_2)$ of the pole position function is a continuous, single-valued function of the control parameters.

When the system has an isolated doublet of resonances which may become degenerate, the corresponding two branches of the pole position function, say $k_n(x_1, x_2)$ and $k_{n+1}(x_1, x_2)$, may be equal (cross or coincide) for some special values of the control parameters. In this case, it is not always possible to solve Eq. (2) for each individual branch without ambiguity and one should proceed to solve Eq. (2) for the pole position function of the two members of the isolated doublet of resonances.

To be precise, let us suppose that there is a finite bounded and connected region $\mathcal{M}$ in parameter space and a finite domain $\mathcal{D}$ in the fourth quadrant of the complex k-plane, such that, when $(x_1, x_2) \in \mathcal{M}$, the Jost function has two and only two zeroes, k_n and k_{n+1}, in the finite domain $\mathcal{D} \in \mathbf{C}$, all other zeroes of $f(-k; x_1, x_2)$

lying outside $\mathcal{D}$. Then, we say that the physical system has an isolated doublet of resonances. To make this situation explicit, the two zeroes of $f(-k; x_1, x_2)$, corresponding to the isolated doublet of resonances may be explicitly factorized as

$$f(-k; x_1, x_2) = \left[\left(k - \frac{1}{2}(k_n + k_{n+1})\right)^2 - \frac{1}{4}(k_n - k_{n+1})^2\right] \times g_{n,n+1}(k, x_1, x_2). \tag{5}$$

When the physical system moves in parameter space from the ordinary point (x_1, x_2) to the exceptional point (x_1^*, x_2^*), the two simple zeroes, $k_n(x_1, x_2)$ and $k_{n+1}(x_1, x_2)$ coalesce into one double zero $k_d(x_1^*, x_2^*)$ of the Jost function,

$$f(-k; x_1^*, x_2^*) = (k - k_d(x_1^*, x_2^*))^2 g_{n,n+1}(k; x_1^*, x_2^*). \tag{6}$$

Equation (5) may also be written as

$$\left[k - \frac{1}{2}(k_n(x_1, x_2) + k_{n+1}(x_1, x_2))\right]^2 - \frac{1}{4}(k_n(x_1, x_2) - k_{n+1}(x_1, x_2))^2 = \frac{f(-k; x_1, x_2)}{g_{n,n+1}(k; x_1, x_2)}, \tag{7}$$

when the external parameters take values in a small neighborhood of the exceptional point $(x_1^*, x_2^*) \in \mathcal{M}$ and $k \in \mathcal{D}$, we may write

$$g_{n,n+1}(k; x_1, x_2) \approx g_{n,n+1}(k_d, x_1^*, x_2^*). \tag{8}$$

Then,

$$\left[k - \frac{1}{2}(k_n(x_1, x_2) + k_{n+1}(x_1, x_2))\right]^2 - \frac{1}{4}(k_n(x_1, x_2) - k_{n+1}(x_1, x_2))^2 \approx \frac{f(-k; x_1, x_2)}{g_{n,n+1}(k_d; x_1^*, x_2^*)}, \tag{9}$$

the term $[g_{n,n+1}(k_d; x_1^*, x_2^*)]^{-1}$ multiplying $f(-k; x_1, x_2)$ may be understood as a finite, non-vanishing, constant scaling factor.

Hence, the qualitative features of the function

$$\bar{f}_{\text{doub}}(-k; x_1, x_2) = \left[k - \frac{1}{2}(k_n(x_1, x_2) + k_{n+1}(x_1, x_2))\right]^2 - \frac{1}{4}(k_n(x_1, x_2) - k_{n+1}(x_1, x_2))^2, \tag{10}$$

and the Jost function $f(-k; x_1, x_2)$, as functions of (x_1, x_2), are the same close to the exceptional point $(x_1^*, x_2^*) \epsilon \mathcal{M}$ for $k \epsilon \mathcal{D}$. More formally, the Jost function

$f(-k; x_1, x_2)$ and the function $\bar{f}_{\text{doub}}(-k; x_1, x_2)$ are contact equivalent at the exceptional point (x_1^*, x_2^*) (Seydel, 1991).

The vanishing of the Jost function defines implicitly, the pole position function $k_{n,n+1}(x_1, x_2)$ of the isolated doublet of resonances

$$\left[\left(k_{n,n+1} - \frac{1}{2}(k_n + k_{n+1})\right)^2 - \frac{1}{4}(k_n - k_{n+1})^2\right] g_{n,n+1}(k_{n,n+1}, x_1, x_2) = 0. \quad (11)$$

If we restrict the external parameters to take values in the region $\mathcal{M}$ in parameter space, the factor $g_{n,n+1}(k_{n,n+1}; x_1, x_2)$ does not vanish when $k_{n,n+1} \in \mathcal{D}$, and it may be canceled in Eq. (11).

Solving for $k_{n,n+1}$, we get

$$k_{n,n+1}(x_1, x_2) = \frac{1}{2}(k_n(x_1, x_2) + k_{n+1}(x_1, x_2)) + \sqrt{\frac{1}{4}(k_n(x_1, x_2) - k_{n+1}(x_1, x_2))^2} \quad (12)$$

with $(x_1, x_2) \in \mathcal{M}$. Since the argument of the square-root function is complex, it is necessary to specify the branch. Here and thereafter, the square root of any complex quantity F will be defined by

$$\sqrt{F} = |\sqrt{F}| \exp\left(i\frac{1}{2}\arg F\right), \quad 0 \le \arg F \le 2\pi \quad (13)$$

so that $|\sqrt{F}| = \sqrt{|F|}$ and the F-plane is cut along the positive real axis. This definition specifies the first branch of $k_{n,n+1}(x_1, x_2)$, the second branch of $k_{n,n+1}(x_1, x_2)$ is given by an expression similar to (12) but with a phase factor $\exp(i\pi)$ in front of the square root in the right hand side of Eq. (12).

Equation (12) relates the pole position function of the doublet of resonances to the pole position functions of the individual resonance states in the doublet. When the two zeroes, k_n and k_{n+1}, coincide exactly, the Jost function has one double zero at

$$k_d = k_n(x_1^*, x_2^*) = k_{n+1}(x_1^*, x_2^*) \quad (14)$$

with $k_d \in \mathcal{D}$ in the fourth quadrant of the complex k-plane.

It is interesting to notice that, the pole position function of the isolated doublet of resonances given in Eqs. (12) and (13), implicitly defined by the vanishing of the Jost function, could also have been obtained from the vanishing of the function $\bar{f}_{\text{doub}}(-k; x_1, x_2)$.

Therefore, the pole position function of the doublet $k_{n,n+1}(x_1, x_2)$, as written in Eq. (12), is contact equivalent (Seydel, 1991) to the multivalued pole position function of the isolated doublet of resonances

$$k_{n,n+1}(x_1, x_2) = f^{-1}(0; x_1, x_2) \tag{15}$$

implicitly defined by the conditions,

$$f(-k_{n,n+1}; x_1, x_2) = 0, \tag{16}$$

and

$$\left(\frac{df(-k; x_1, x_2)}{dk}\right)_{k_d} = 0, \tag{17}$$

$$\left(\frac{df^2(-k; x_1, x_2)}{dk^2}\right)_{k_d} \neq 0, \tag{18}$$

for (x_1, x_2) in a neighborhood of the exceptional point $(x_1^*, x_2^*) \epsilon \mathcal{M}$ and $k \epsilon \mathcal{D}$.

2.1. The Analytical Behavior of the Pole-Position Function at the Exceptional Point

The contact equivalence of the two expressions for the pole position function of the isolated doublet of resonances, Eqs. (12) and (15)–(18), will allow us to determine the nature of the singularity of this function at the crossing of resonances in parameter space.

We will start by showing that the derivatives of the functions $1/2(k_n(x_1, x_2) + k_{n+1}(x_1, x_2))$ and $1/4(k_n(x_1, x_2) - k_{n+1}(x_1, x_2))^2$ are finite at the exceptional point. Then, a Taylor series expansion of $1/4\,(k_n(x_1, x_2) - k_{n+1}(x_1, x_2))^2$ at the exceptional point (x_1^*, x_2^*) will give us the analytical behavior of $k_{n,n+1}(x_1, x_2)$ in the neighborhood of the crossing of resonances, as function of (x_1, x_2).

The derivatives of $k_n(x_1, x_2)$ may be computed from the Jost function written as (Krantz and Parks, 2002),

$$f(-k; x_1, x_2) = (k - k_n(x_1, x_2))\,(k - k_{n+1}(x_1, x_2))\,g_{n,n+1}(k; x_1, x_2) \tag{19}$$

then,

$$\left[\left(\frac{\partial f(-k; x_1, x_2)}{\partial x_i}\right)_{x_{i+1}}\right]_{k=k_n} = -\left(\frac{\partial k_n(x_1, x_2)}{\partial x_i}\right)_{x_{i+1}} (k_n - k_{n+1})\,g_{n,n+1}(k_n; x_1, x_2) \tag{20}$$

and a similar expression for $[(\partial f(-k; x_1, x_2)/\partial x_i)_{x_{i+1}}]_{k_{n+1}}$ which is obtained from Eq. (20) by exchanging $k_n(x_1, x_2)$ and $k_{n+1}(x_1, x_2)$. Notice that this exchange changes the sign in the right hand side of Eq. (20). Adding the two derivatives and rearranging some terms we get

$$\left[\left(\frac{\partial f(-k;x_1,x_2)}{\partial x_1}\right)_{x_2}\right]_{k=k_n}+\left[\left(\frac{\partial f(-k;x_1,x_2)}{\partial x_1}\right)_{x_2}\right]_{k=k_{n+1}}$$
$$=-\frac{1}{4}\left(\frac{\partial\,(k_n(x_1,x_2)-k_{n+1}(x_1,x_2))^2}{\partial x_1}\right)_{x_2}$$
$$\times(g_{n,n+1}(k_n;x_1,x_2)+g_{n,n+1}(k_{n+1};x_1,x_2)) \tag{21}$$
$$-\frac{1}{2}\,(k_n(x_1,x_2)-k_{n+1}(x_1,x_2))\left(\frac{\partial\,(k_n(x_1,x_2)+k_{n+1}(x_1,x_2))}{\partial x_1}\right)_{x_2}$$
$$\times(g_{n,n+1}(k_n;x_1,x_2)-g_{n,n+1}(k_{n+1};x_1,x_2))$$

and a similar expression for the derivatives with respect to x_2.

When we take the limit as (x_1, x_2) goes to (x_1^*, x_2^*), and, recalling that in this limit $k_n(x_1^*, x_2^*) = k_{n+1}(x_1^*, x_2^*) = k_d$, we get

$$\left[\left(\frac{\partial\,(k_n(x_1,x_2)-k_{n+1}(x_1,x_2))^2}{\partial x_1}\right)_{x_2^*}\right]_{k=k_d}$$
$$=\frac{-4}{g_{n,n+1}(k_d;x_1^*,x_2^*)}\left[\left(\frac{\partial f(-k;x_1,x_2)}{\partial x_1}\right)_{x_2^*}\right]_{k=k_d}. \tag{22}$$

The term $g_{n,n+1}(k_d; x_1^*, x_2^*)$ which appears in the denominator of the right hand side of Eq. (22) may also be expressed in terms of derivatives of the Jost function. From Eq. (19), a straightforward computation gives

$$\left[\left(\frac{\partial^2 f(-k;x_1,x_2)}{\partial k^2}\right)\right]_{k=k_d}=2g_{n,n+1}(k_d;x_1^*,x_2^*). \tag{23}$$

Substitution of this expression in Eq. (22), gives

$$\left[\left(\frac{\partial\,(k_n(x_1,x_2)-k_{n+1}(x_1,x_2))^2}{\partial x_1}\right)_{x_2}\right]_{k=k_d}$$
$$=\frac{-8}{\left[\left(\frac{\partial^2 f(-k;x_1,x_2)}{\partial k^2}\right)_{x_1^*,x_2^*}\right]_{k=k_d}}\left[\left(\frac{\partial f(-k;x_1,x_2)}{\partial x_1}\right)_{x_2}\right]_{k=k_d}. \tag{24}$$

Notice that the function $1/4\,(k_n(x_1,x_2)-k_{n+1}(x_1,x_2))^2$ vanishes at the exceptional point (x_1, x_2), but its first derivatives with respect to the external parameters at that point are finite and non-vanishing.

However, the derivatives of the function $1/2\,(k_n(x_1,x_2)-k_{n+1}(x_1,x_2))$ do not exist at the exceptional point $(x_1^*, x_2^*)\in\mathcal{M}$. From Eq. (24)

$$\left[\left(\frac{\partial\frac{1}{2}\left(k_n(x_1,x_2)-k_{n+1}(x_1,x_2)\right)}{\partial x_i}\right)_{x_{i+1}}\right]_{k=k_d} = \frac{1}{\left(k_n(x_1,x_2)-k_{n+1}(x_1,x_2)\right)} \times \frac{-4}{\left[\left(\frac{\partial^2 f(-k;x_1,x_2)}{\partial k^2}\right)_{x_1^*,x_2^*}\right]_{k=k_d}}\left[\left(\frac{\partial^2 f(-k;x_1,x_2)}{\partial x_i}\right)_{x_{i+1}}\right]_{k=k_d} \tag{25}$$

the right hand side of this expression tends to infinity as (x_1,x_2) tends to (x_1^*,x_2^*) and $k_n(x_1,x_2)-k_{n+1}(x_1,x_2)$ vanishes.

A similar computation will give us the derivatives of $1/2(k_n(x_1,\ x_2)+k_{n+1}(x_1,x_2))$ with respect to the external parameters. From Eq. (20), we get

$$\begin{aligned}&\left[\left(\frac{\partial f(-k;x_1,x_2)}{\partial x_1}\right)_{x_2}\right]_{k=k_n} - \left[\left(\frac{f(-k;x_1,x_2)}{\partial x_1}\right)_{x_2}\right]_{k=kn+1}\\ &= \left(k_{n+1}(x_1,x_2)-k_n(x_1,x_2)\right)\left(\frac{\partial\left(k_n(x_1,x_2)+k_{n+1}(x_1,x_2)\right)}{\partial x_1}\right)_{x_2}\\ &\quad\times\frac{1}{2}\left(g_{n,n+1}(k_n;x_1,x_2)+g_{n,n+1}(k_{n+1};x_1,x_2)\right)\\ &\quad-\frac{1}{2}\left(\frac{\partial\left(k_n(x_1,x_2)-k_{n+1}(x_1,x_2)\right)^2}{\partial x_1}\right)_{x_2}\\ &\quad\times\frac{1}{2}\left(g_{n,n+1}(k_n,x_1,x_2)-\ g_{n,n+1}(k_{n+1},x_1,x_2)\right)\end{aligned} \tag{26}$$

Solving for $[\partial\left(k_n(x_1,x_2)+k_{n+1}(x_1,x_2)\right)/\partial x_1]_{x_2}$, we obtain

$$\begin{aligned}&\left(\frac{\partial\left(k_n(x_1,x_2)+k_{n+1}(x_1,x_2)\right)}{\partial x_1}\right)_{x_2} = \frac{2}{g_{n,n+1}(k_n;x_1,x_2)+g_{n,n+1}(k_{n+1},x_1,x_2)}\\ &\times\Big\{\left[k_{n+1}(x_1,x_2)-k_n(x_1,x_2)\right]^{-1}\\ &\times\left[\left(\frac{\partial f(-k;x_1,x_2)}{\partial x_i}\right)_{x_2,k=k_n}-\left(\frac{\partial f(-k;x_1,x_2)}{\partial x_1}\right)_{x_2,k=k_{n+1}}\right]\\ &+\frac{1}{2}\left(\frac{\partial\left(k_n(x_1,x_2)-k_{n+1}(x_1,x_2)\right)^2}{\partial x_1}\right)_{x_2}\\ &\times\ \frac{1}{2}\frac{\left(g_{n,n+1}(k_n;x_1,x_2)-g_{n,n+1}(k_{n+1},x_1,x_2)\right)}{k_{n+1}(x_1,x_2)-k_n(x_1,x_2)}\Big\}.\end{aligned} \tag{27}$$

If we take the limit as (x_1, x_2) goes to (x_1^*, x_2^*), $k_n(x_1^*, x_2^*) = k_{n+1}(x_1^*, x_2^*) = k_d(x_1^*, x_2^*)$, we get

$$\left[\left(\frac{\partial\,(k_n(x_1,x_2)+k_{n+1}(x_1,x_2))}{\partial x_1}\right)_{x_2}\right]_{k=k_d} = \frac{1}{g_{n,n+1}(k_d; x_1^*, x_2^*)}$$
$$\times\left\{-\left(\frac{\partial^2 f(-k;x_1,x_2)}{\partial x_1,\partial k}\right)_{x_2^*,k=k_d} - \frac{1}{2}\left(\frac{\partial\,(k_n(x_1,x_2)-k_{n+1}(x_1,x_2))^2}{\partial x_1}\right)_{x_2^*}\right.$$
$$\left.\times\ \frac{1}{2}\left(\frac{\partial g_{n,n+1}(k;x_1,x_2)}{\partial k}\right)_{k=k_d}\right\}. \tag{28}$$

Now from Eq. (5),

$$\left[6\left(\frac{\partial g_{n,n+1}(k,x_1,x_2)}{\partial k}\right)_{(x_1^*,x_2^*)}\right]_{k_d} = \left[\left(\frac{\partial^3 f(-k;x_1,x_2)}{\partial k^3}\right)_{x_1^*,x_2^*}\right]_{k_d}. \tag{29}$$

When we substitute this expression and (23) and (24) for $g_{n,n+1}(k_d; x_1^*, x_2^*)$ and $[\partial\,(k_n(x_1,x_2)-k_{n+1}(x_1,x_2))^2/\partial x_1]_{x_2,k_d}$ in Eq. (28), and rearranging some terms, we finally get

$$\frac{1}{2}\left[\left(\frac{\partial\,(k_n(x_1,x_2)+k_{n+1}(x_1,x_2))}{\partial x_1}\right)_{x_2}\right]_{k=k_d} = \frac{-1}{\left[\left(\frac{\partial^2 f(-k;x_1,x_2)}{\partial k^2}\right)_{x_1^*,x_2^*}\right]_{k=d_d}}$$
$$\times\left\{\left[\left(\frac{\partial^2 f(-k;x_1,x_2)}{\partial x_1,\partial k}\right)_{(x_1^*,x_2^*)}\right]_{k=k_d} - \frac{1}{\left[\left(\frac{\partial^2 f(-k;x_1,x_2)}{\partial k^2}\right)_{(x_1^*,x_2^*)}\right]_{k=k_d}}\right.$$
$$\left.\times\ \frac{1}{3}\left[\left(\frac{\partial^3 f(-k;x_1,x_2)}{\partial k^3}\right)_{(x_1^*,x_2^*)}\right]_{k=k_d}\left[\left(\frac{\partial f(-k;x_1,x_2)}{\partial x_1}\right)_{(x_1^*,x_2^*)}\right]_{k=k_d}\right\} \tag{30}$$

and a similar expression for $[\partial\,(k_n(x_1,x_2)+k_{n+1}(x_1,x_2))/\partial x_2]_{x_1}$.

In this way, we have shown that the first derivatives of the functions $1/2\,[k_n(x_1,x_2)+k_{n+1}(x_1,x_2)]$ and $1/4\,[k_n(x_1,x_2)-k_{n+1}(x_1,x_2)]^2$ with respect to the external parameters are finite and non-vanishing at the exceptional point (x_1^*, x_2^*).

From these results, the first terms in a Taylor series expansion of the functions $1/2\,(k_n(x_1,x_2)+k_{n+1}(x_1,x_2))$ and $1/4\,(k_n(x_1,x_2)-k_{n+1}(x_1,x_2))^2$ about the exceptional point (x_1^*, x_2^*) are

$$\frac{1}{2}(k_n(x_1,x_2)+k_{n+1}(x_1,x_2)) = k_d + \sum_{i=1}^{2} d_i^{(1)}(x_i - x_i^*) + O\left((x_i - x_i^*)^2\right) \tag{31}$$

and

$$(k_n(x_1,x_2)-k_{n+1}(x_1,x_2))^2 = \sum_{i=1}^{2} c_i^{(1)}(x_1 - x_i^*) + O((x_i - x_i^*)^2). \tag{32}$$

The derivatives $c_i^{(1)}$ and $d_i^{(1)}$, expressed in terms of derivatives of the Jost function at the degeneracy point, are given in Eqs. (24) and (30), respectively.

Substitution of these expressions in Eq. (12) gives a simple but accurate representation of the analytical behavior of the wave number-pole position function as function of the control parameters

$$k_{n,n+1}(x_1,x_2) \approx \hat{k}_{n,n+1}(x_1,x_2) \tag{33}$$

where

$$\hat{k}_{n,n+1}(x_1,x_2) = k_d + \Delta k_d(x_1,x_2) + \sqrt{\frac{1}{4}\left[c_1^{(1)}(x_1 - x_1^*) + c_2^{(1)}\left(x_2 - x_2^{(*)}\right)\right]} \tag{34}$$

for (x_1, x_2) in a small neighborhood of the exceptional point (x_1^*, x_2^*).

Energy-pole position function: this result may readily be translated into a similar assertion for the resonance energy eigenvalues, $\mathcal{E}_n(x_1,x_2)$ and $\mathcal{E}_{n+1}(x_1,x_2)$, of the isolated doublet of resonances.

Let us take the square of both sides of Eq. (12), multiplying them by $\hbar^2/2m$ and recalling Eq. (1), we get

$$\mathcal{E}_{n,n+1}(x_1,x_2) = \frac{1}{2}\left(\mathcal{E}_n(x_1,x_2)+\mathcal{E}_{n+1}(x_1,x_2)\right) + \sqrt{\frac{1}{4}\left(\mathcal{E}_n(x_1,x_2)-\mathcal{E}_{n+1}(x_1,x_2)\right)^2} \tag{35}$$

where

$$\frac{1}{2}\left(\mathcal{E}_n(x_1,x_2)+\mathcal{E}_{n+1}(\xi_1,\xi_2)\right)=\frac{\hbar^2}{2m}\left[\frac{1}{4}\left(k_n(x_1,x_2)+k_{n+1}(x_1,x_2)\right)^2 + \frac{1}{4}\left(k_n(x_1,x_2)-k_{n+1}(x_1,x_2)\right)^2\right]. \quad (36)$$

The first two terms in square brackets in the right hand side of this equation are regular functions of (x_1, x_2) at the exceptional point, and may be expanded in a Taylor series about (x_1^*, x_2^*) according to Eqs. (31) and (32). The term under the square root in the right hand side of Eq. (35) is also a regular function of the external parameters at the exceptional point, since it is the product of two regular functions,

$$\frac{1}{4}\left(\mathcal{E}_n(x_1,x_2)-\mathcal{E}_{n+1}(x_1,x_2)\right)^2=\left(\frac{\hbar^2}{2m}\right)^2\frac{1}{4}\left(k_n(x_1,x_2)-k_{n+1}(x_1,x_2)\right)^2 \times \left(k_n(x_1,x_2)+k_{n+1}(x_1,x_2)\right)^2. \quad (37)$$

However, the difference of the two complex resonance energy eigenvalues,

$$\frac{1}{2}\left(\mathcal{E}_n(x_1.x_2)-\mathcal{E}_{n+1}(x_1,x_2)\right)=\left(\frac{\hbar^2}{2m}\right)\left(k_n(x_1,x_2)\ +k_{n+1}(x_1,x_2)\right) \times\frac{1}{2}\left(k_n(x_1,x_2)-k_{n+1}(x_1,x_2)\right), \quad (38)$$

is not a regular function of the external parameters since the derivatives of $1/2\,(k_n(x_1,x_2)-k_{n+1}(x_1,x_2))$ do not exist at the exceptional point.

Therefore, the behavior of the complex resonance energy eigenvalues $\mathcal{E}_n(x_1,x_2)$ and $\mathcal{E}_{n+1}(x_1,x_2)$ as functions of the control parameters (x_1,x_2) close to the exceptional point (x_1^*, x_2^*) is obtained from the expression

$$\mathcal{E}_{n,n+1}(x_1,x_2)\approx\hat{\mathcal{E}}_{n,n+1}(x_1,x_2), \quad (39)$$

$$\hat{\mathcal{E}}_{n,n+1}(x_1,x_2)=\mathcal{E}_d+\Delta\mathcal{E}_d+\sqrt{\frac{1}{4}\left[C_1^{(1)}(x_1-x_1^*)+C_2^{(1)}(x_2-x_2^*)\right]}, \quad (40)$$

where

$$C_i^{(1)}=\left(\frac{\hbar^2 k_d}{m}\right)^2 c_i^{(1)}. \quad (41)$$

To examine the precise nature of the singularity of the energy-pole position function $\mathcal{E}_{n,n+1}(x_1,x_2)$, as a function of the real parameters (x_1,x_2), it will be convenient to fix the origin of coordinates at the exceptional point and change slightly the notation.

We introduce three vectors in parameter space

$$\vec{\xi} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_1^* \\ x_2 - x_2^* \end{pmatrix}, \tag{42}$$

$$\vec{R} = \begin{pmatrix} \text{Re } C_1^{(1)} \\ \text{Re } C_2^{(1)} \end{pmatrix} \tag{43}$$

and

$$\vec{I} = \begin{pmatrix} \text{Im } C_1^{(1)} \\ \text{Im } C_2^{(1)} \end{pmatrix}. \tag{44}$$

The components of the real fixed vectors $\vec{R}$ and $\vec{I}$ are the real and imaginary parts of the coefficients $C_i^{(1)}$ of $(x_i - x_i^*)$ in the Taylor expansion of the function $1/4\,(\mathcal{E}_n(x_1, x_2) - \mathcal{E}_{n+1}(x_1, x_2))^2$ and the real vector $\vec{\xi}$ is the position vector of the point (x_1, x_2) relative to the exceptional point (x_1^*, x_2^*) in parameter space.

Let us call $\epsilon_{n,n+1}^2(x_1, x_2)$ the term which appears under the square root in the right hand side of Eq. (40)

$$\epsilon_{n+1}^2(x_1, x_2) = \frac{1}{4}(\mathcal{E}_n(x_1, x_2) - \mathcal{E}_{n,n+1}(x_1, x_2))^2 \tag{45}$$

and let $\hat{\epsilon}_{n,n+1}^2(x_1, x_2)$ be the first order term in the Taylor expansion of $1/4\,(\mathcal{E}_n(x_1, x_2) - \mathcal{E}_{n+1}(x_1, x_2))^2$ about (x_1^*, x_2^*), which appears under the square root in the right hand side of Eq. (40). Then,

$$\epsilon_{n,n+1} \approx \hat{\epsilon}_{n,n+1}(x_1, x_2), \tag{46}$$

where

$$\hat{\epsilon}_{n,n+1}(x_1, x_2) = \sqrt{\frac{1}{4}\left[C_1^{(1)}(x_1 - x_1^*) + C_2^{(1)}(x_2 - x_2^*)\right]}, \tag{47}$$

$$\mathcal{E}_{n,n+1}(x_1, x_2) \approx \mathcal{E}_d + \Delta\mathcal{E}_d(x_1, x_2) + \hat{\epsilon}_{n,n+1}(x_1, x_2), \tag{48}$$

or, in the notation defined in Eqs. (42)–(44),

$$\hat{\epsilon}_{n,n+1}^2(x_1, x_2) = \frac{1}{4}((\vec{R} \cdot \vec{\xi}) + i(\vec{I} \cdot \vec{\xi})) \tag{49}$$

and

$$|\hat{\epsilon}_{n,n+1}(x_1, x_2)|^2 = +\sqrt{\frac{1}{4}((\vec{R} \cdot \vec{\xi})^2 + (\vec{I} \cdot \vec{\xi})^2)}. \tag{50}$$

From the identities

$$(\text{Re } \epsilon)^2 = \frac{1}{4}(\epsilon^2 + \epsilon^{*2} + 2\epsilon\epsilon^*) = \frac{1}{2}(\text{Re } (\epsilon^2) + |\epsilon|^2) \tag{51}$$

and

$$(\mathrm{Im}\,\epsilon)^2 = \frac{1}{2}(|\epsilon|^2 - \mathrm{Re}\,(\epsilon^2)), \tag{52}$$

and Eqs. (47)–(49), we obtain

$$(\mathrm{Re}\,\hat{\epsilon}_{n,n+1}(x_1, x_2))^2 = \frac{1}{8}\left[+\sqrt{(\vec{R}\cdot\vec{\xi})^2 + (\vec{I}\cdot\vec{\xi})^2} + \vec{R}\cdot\vec{\xi}\right] \tag{53}$$

and

$$(\mathrm{Im}\,\hat{\epsilon}_{n,n+1}(x_1, x_2))^2 = \frac{1}{8}\left[+\sqrt{(\vec{R}\cdot\vec{\xi})^2 + (\vec{I}\cdot\vec{\xi})^2} - \vec{R}\cdot\vec{\xi}\right]. \tag{54}$$

Since the square root term inside the brackets is positive, the functions $(\mathrm{Re}\,\hat{\epsilon}_{n,n+1}(x_1, x_2))^2$ and $(\mathrm{Im}\,\hat{\epsilon}_{n,n+1}(x_1, x_2))^2$ are real, positive, single-valued functions of (ξ_1, ξ_2).

Therefore, the real and imaginary parts of the function $\hat{\epsilon}_{n,n+1}(x_1, x_2)$ are

$$\mathrm{Re}\,\hat{\epsilon}_{n,n+1}(x_1, x_2) = \pm\frac{1}{2\sqrt{2}}\left[+\sqrt{(\vec{R}\cdot\vec{\xi})^2 + (\vec{I}\cdot\vec{\xi})^2} + \vec{R}\cdot\vec{\xi}\right]^{1/2}, \tag{55}$$

$$\mathrm{Im}\,\hat{\epsilon}_{n,n+1}(x_1, x_2) = \pm\frac{1}{2\sqrt{2}}\left[+\sqrt{(\vec{R}\cdot\vec{\xi})^2 + (\vec{I}\cdot\vec{\xi})^2} - \vec{R}\cdot\vec{\xi}\right]^{1/2}. \tag{56}$$

Now, from the Eq. (49) we obtain

$$\mathrm{sign}\,(\mathrm{Re}\,\hat{\epsilon}_{n,n+1}(\xi_1, \xi_2))\ \mathrm{sign}\,(\mathrm{Im}\,\hat{\epsilon}_{n,n+1}(\xi_1, \xi_2)) = \mathrm{sign}\,(\vec{I}\cdot\vec{\xi}). \tag{57}$$

It follows from Eq. (55), that close to the exceptional point (origin of coordinates), $\mathrm{Re}\,\hat{\epsilon}_{n,n+1}(x_1, x_2)$ is a two branched function of (ξ_1, ξ_2) which may be represented as a two-sheeted surface S_R, in a three dimensional Euclidean space with Cartesian coordinates $(\mathrm{Re}\,\hat{\epsilon}_{n,n+1}, \xi_1, \xi_2)$. The two branches of $\mathrm{Re}\,\hat{\epsilon}_{n,n+1}(\xi_1, \xi_2)$ are represented by two sheets which are copies of the plane (ξ_1, ξ_2) cut along a line where the two branches of the function are joined smoothly. Since a negative and a positive numbers are equal only when both vanish, the cut is defined as the locus of the points where the argument of the square root function in the right hand side of Eq. (55) vanishes.

Close to the origin of coordinates (the exceptional point), this locus is defined by a unit vector $\hat{\xi}_c$ in the $(\vec{\xi}_1, \vec{\xi}_2)$, plane such that

$$\vec{I}\cdot\hat{\xi}_c = 0 \quad \text{and} \quad \vec{R}\cdot\hat{\xi}_c = -|\vec{R}\cdot\hat{\xi}_c| \tag{58}$$

Therefore, the real part of the energy-pole position function,

$\mathrm{Re}\,\mathcal{E}_{n,n+1}(x_1, x_2)$, as a function of the real parameters (x_1, x_2), has an algebraic branch point of square root type (rank one) at the exceptional point with coordinates (x_1^*, x_2^*) in parameter space, and a branch cut along a line, $\mathcal{L}_R$, that starts at the exceptional point and extends in the *positive* direction defined by the unit vector $\hat{\xi}_c$ satisfying Eq. (58).

A similar analysis shows that, the imaginary part of the energy-pole position function, Im $\mathcal{E}_{n,n+1}(x_1, x_2)$, as a function of the real parameters (x_1, x_2), also has an algebraic branch point of square root type (rank one) at the exceptional point with coordinates (x_1^*, x_2^*) in parameter space, and also has a branch cut along a line, $\mathcal{L}_I$, that starts at the exceptional point and extends in the *negative* direction defined by the unit vector $\hat{\xi}_c$ satisfying Eq. (58).

The branch cut lines, $\mathcal{L}_R$ and $\mathcal{L}_I$, are orthogonal to each other – they are in orthogonal subspaces of a four dimensional Euclidean space with coordinates (Re $\epsilon_{n,n+1}$, Im $\epsilon_{n,n+1}$, ξ_1, ξ_2) – but have one point in common, the exceptional point with coordinate (x_1^*, x_2^*).

Along the line $\mathcal{L}_R$, excluding the exceptional point (x_1^*, x_2^*),

$$\text{Re } \mathcal{E}_n(x_1, x_2) = \text{Re } \mathcal{E}_{n+1}(x_1, x_2), \tag{59}$$

but

$$\text{Im } \mathcal{E}_n(x_1, x_2) \neq \text{Im } \mathcal{E}_{n+1}(x_1, x_2). \tag{60}$$

Similarly, along the line $\mathcal{L}_I$, excluding the exceptional point,

$$\text{Im } \mathcal{E}_n(x_1, x_2) = \text{Im } \mathcal{E}_{n+1}(x_1, x_2), \tag{61}$$

but

$$\text{Re } \mathcal{E}_n(x_1, x_2) \neq \text{Re } \mathcal{E}_{n+1}(x_1, x_2). \tag{62}$$

Equality of the complex resonance energy eigenvalues (degeneracy of resonances),

$$\mathcal{E}_n(x_1^*, x_2^*) = \mathcal{E}_{n+1}(x_1^*, x_2^*) = \mathcal{E}_d \tag{63}$$

occurs only at the exceptional point with coordinates (x_1^*, x_2^*) in parameter space and only at that point.

In consequence, in the complex energy plane, the crossing point of two simple resonance poles of the scattering matrix is an isolated point where the scattering matrix has one double resonance pole.

Let us end this section with the following remark: in the general case, a variation of the vector of parameters causes a perturbation of the energy eigenvalues. In the particular case of a double complex resonance energy eigenvalue $\mathcal{E}_d$, associated with a chain of length two of generalized Jordan-Gamow eigenfunctions, we are considering here, the perturbation series expansion of the eigenvalues $\mathcal{E}_n$, $\mathcal{E}_{n+1}$ about $\mathcal{E}_d$ in terms of the small parameter $|\xi|$, Eqs. (48)–(49), takes the form of a Puiseux series

$$\begin{aligned} \mathcal{E}_{n,n+1}(x_1, x_2) = \mathcal{E}_d + |\xi|^{1/2} \sqrt{\frac{1}{4}[(\vec{R} \cdot \hat{\xi}) + i(\vec{I} \cdot \hat{\xi})]} \\ + \Delta\mathcal{E}_d(x_1, x_2) + O(|\xi|^{3/2}) \end{aligned} \tag{64}$$

with fractional powers $|\xi|^{j/2}$, $j = 0, 1, 2, \ldots$ of the small parameter $|\xi|$ (Kato, 1980).

3. UNFOLDING OF THE DEGENERACY POINT

In Section 2, the pole position function of the isolated doublet of resonances was implicitly defined by the vanishing of the Jost function, Eq. (15), and the conditions (16)–(18). By formally solving those equations, we introduced an explicit expression for the double-valued energy-pole position function, $\mathcal{E}_{n,n+1}(\xi_1, \xi_2)$, in terms of the individual resonance energy eigenvalues of the individual components of the isolated doublet of resonances through Eqs. (12) and (35). However, an explicit association of each individual resonance energy eigenvalue with the branches of the pole position function was not made. The function $\mathcal{E}_{n,n+1}(\xi_1, \xi_2)$ was later approximated by the expressions

$$\mathcal{E}_{n,n+1}(\xi_1, \xi_2) \approx \mathcal{E}_d + \Delta\mathcal{E}_d(\xi_1, \xi_2) + \hat{\epsilon}_{n,n+1}(\xi_1, \xi_2) \tag{65}$$

where, the single valued function

$$\Delta\mathcal{E}_d(\xi_1, \xi_2) \approx \frac{1}{2}\left(\mathcal{E}_n(\xi_1, \xi_2) + \mathcal{E}_{n+1}(\xi_1, \xi_2)\right) - \mathcal{E}_d \tag{66}$$

is explicitly given in Eqs. (38)–(40), and, the double valued function

$$\hat{\epsilon}_{n,n+1}(\xi_1, \xi_2) \approx \pm\sqrt{\frac{1}{4}\left(\mathcal{E}_n(\xi_1, \xi_2) - \mathcal{E}_{n+1}(\xi_1, \xi_2)\right)^2} \tag{67}$$

is explicitly given in Eqs. (55) and (56).

In this section we will associate the individual resonance energy eigenvalues with the branches of the pole position function $\mathcal{E}_{n,n+1}(\xi_1, \xi_2)$ and its contact equivalent approximation $\hat{\mathcal{E}}_{n,n+1}(\xi_1, \xi_2)$.

Let us start by observing that, the two functions on the right hand side of Eqs. (66) and (67) are invariant under the exchange of the indices n and $n + 1$, showing that these equations by themselves do not determine completely the association of each individual energy eigenvalue with the branches of the pole position function. This is so, because the labeling of the individual resonance energy eigenvalues with the indices n and $n + 1$ is purely a matter of convention.

In order to fix this convention, let us consider a point (ξ_1, ξ_2) in a neighborhood of, but not equal to, the exceptional point, and which is not in any of the two branch cut lines. To this point corresponds a pair of non-degenerate complex resonance energy eigenvalues, labeled $\mathcal{E}_n(\xi_1, \xi_2)$ and $\mathcal{E}_{n+1}(\xi_1, \xi_2)$. Then, close to the exceptional point, the energy eigenvalues and the function $\hat{\epsilon}_{n,n+1}(\xi_1, \xi_2)$ will, initially and conventionally, be related by

$$\hat{\epsilon}_{n,n+1}(\xi_1, \xi_2) \approx \frac{1}{2}\left(\mathcal{E}_n(\xi_1, \xi_2) - \mathcal{E}_{n+1}(\xi_1, \xi_2)\right). \tag{68}$$

This relation is the convention that was missing in Eqs. (12)–(13) and (36)–(37) to completly define the association of each individual energy eigenvalue with the energy-pole position function. However, once this convention is fixed, it determines the way in which the function Re $\epsilon_{n,n+1}(\xi_1, \xi_2)$ and Im $\epsilon_{n,n+1}(\xi_1, \xi_2)$ are initially related to the energy eigenvalues.

Therefore, from (67) and (68), when Re $\mathcal{E}_n(\xi_1, \xi_2) >$ Re $\mathcal{E}_{n+1}(\xi_1, \xi_2)$ we get

$$\text{Re}\ \mathcal{E}_n(\xi_1, \xi_2) = \frac{1}{2}\text{Re}\ (\mathcal{E}_n(\xi_1, \xi_2) + \mathcal{E}_{n+1}(\xi_1, \xi_2)) + \text{Re}\ \epsilon^{(+)}_{n,n+1}(\xi_1, \xi_2), \tag{69}$$

and

$$\text{Re}\ \mathcal{E}_{n+1}(\xi_1, \xi_2) = \frac{1}{2}\text{Re}\ (\mathcal{E}_n(\xi_1, \xi_2) + \mathcal{E}_{n+1}(\xi_1, \xi_2)) + \text{Re}\ \epsilon^{(-)}_{n,n+1}(\xi_1, \xi_2). \tag{70}$$

In these expressions $\epsilon^{(+)}_{n,n+1}$ is the positive branch of $\epsilon_{n,n+1}$ and $\epsilon^{(-)}_{n,n+1}$ is the negative branch.

When Re $\mathcal{E}_n(\xi_1, \xi_2) <$ Re $\mathcal{E}_{n+1}(\xi_1, \xi_2)$, the branch signature labels $(+)$ and $(-)$ on $\epsilon_{n,n+1}(\xi_1, \xi_2)$ in the right hand side of Eqs. (69) and (70) are exchanged.

The imaginary parts of the energy eigenvalues, Im $\mathcal{E}_n(\xi_1, \xi_2)$ and Im $\mathcal{E}_{n+1}(\xi_1, \xi_2)$, are associated with the branches of the real function Im $\epsilon_{n,n+1}(\xi_1, \xi_2)$ according to a similar rule.

When Im $\mathcal{E}_{n+1}(\xi_1, \xi_2) >$ Im $\mathcal{E}_n(\xi_1, \xi_2)$, we get

$$\text{Im}\ \mathcal{E}_n(\xi_1, \xi_2) = \text{Im}\ \frac{1}{2}\ (\mathcal{E}_n(\xi_1, \xi_2) + \mathcal{E}_{n+1}(\xi_1, \xi_2)) + \text{Im}\ \epsilon^{(-)}_{n,n+1}(\xi_1, \xi_2) \tag{71}$$

and

$$\text{Im}\ \mathcal{E}_{n+1}(\xi_1, \xi_2) = \text{Im}\ \frac{1}{2}\ (\mathcal{E}_n(\xi_1, \xi_2) + \mathcal{E}_{n+1}(\xi_1, \xi_2)) + \text{Im}\ \epsilon^{(+)}_{n,n+1}(\xi_1, \xi_2). \tag{72}$$

When the point (ξ_1, ξ_2) is on the projection of the branch cut line $\mathcal{L}'_R$,

$$\text{Re}\ \mathcal{E}_n(\xi_1, \xi_2) = \text{Re}\ \mathcal{E}_{n+1}(\xi_1, \xi_2), \tag{73}$$

but

$$\text{Im}\ \mathcal{E}_n(\xi_1, \xi_2) \neq \text{Im}\ \mathcal{E}_{n+1}(\xi_1, \xi_2). \tag{74}$$

Similarly, when the point (ξ_1, ξ_2) is on the projection of the branch cut line $\mathcal{L}'_I$,

$$\text{Im}\ \mathcal{E}_n(\xi_1, \xi_2) = \text{Im}\ \mathcal{E}_{n+1}(\xi_1, \xi_2), \tag{75}$$

but

$$\text{Re}\ \mathcal{E}_n(\xi_1, \xi_2) \neq \text{Re}\ \mathcal{E}_{n+1}(\xi_1, \xi_2). \tag{76}$$

When the point that represents the control parameters of the physical system moves continuously in parameter space along a path that crosses the projection of the branch cut line $\mathcal{L}'_R$, Re $\mathcal{E}_n(\xi_1, \xi_2)$ goes continuously from the positive to the negative branch of Re $\epsilon_{n,n+1}(\xi_1, \xi_2)$, while Re $\mathcal{E}_{n+1}(\xi_1, \xi_2)$ changes smoothly from the negative to the positive branch of Re $\epsilon_{n,n+1}(\xi_1, \xi_2)$. When the point (ξ_1, ξ_2) crosses the line $\mathcal{L}'_R$, the sense of the inequality that defines the branch signature label is inverted, and the rule that associates the real part of the energy eigenvalues, $\mathcal{E}_n(\xi_1, \xi_2)$ and $\mathcal{E}_{n+1}(\xi_1, \xi_2)$, with the branches of the function Re $\epsilon_{n,n+1}(\xi_1, \xi_2)$ is still satisfied. A similar reasoning shows that when the point (ξ_1, ξ_2) moves continuously along a path that crosses the projection of the branch cut line $\mathcal{L}'_I$, the rule that associates the imaginary parts of the energy eigenvalues, $\mathcal{E}_n(\xi_1, \xi_2)$ and $\mathcal{E}_{n+1}(\xi_1, \xi_2)$, with the branches of the function Im $\epsilon_{n,n+1}(\xi_1, \xi_2)$ is also satisfied.

It follows from these rules that, close to the exceptional point, the energy eigenvalues are given by the expression

$$\begin{aligned}\hat{\mathcal{E}}_n(\xi_1, \xi_2) = \mathcal{E}_d + \Delta\mathcal{E}_{n,n+1}(\xi_1, \xi_2)& \\ + \sigma_R^{(n)} \frac{1}{2\sqrt{2}}\left[+\sqrt{(\vec{R}\cdot\vec{\xi})^2 + (\vec{I}\cdot\vec{\xi})^2} + (\vec{R}\cdot\vec{\xi})\right]^{1/2}& \\ + i\sigma_I^{(n)} \frac{1}{2\sqrt{2}}\left[+\sqrt{(\vec{R}\cdot\vec{\xi})^2 + (\vec{I}\cdot\vec{\xi})^2} - (\vec{R}\cdot\vec{\xi})\right]^{1/2}&\end{aligned} \tag{77}$$

and

$$\begin{aligned}\hat{\mathcal{E}}_{n+1}(\xi_1, \xi_2) = \mathcal{E}_d + \Delta\mathcal{E}_{n,n+1}(\xi_1, \xi_2)& \\ + \sigma_R^{(n+1)} \frac{1}{2\sqrt{2}}\left[+\sqrt{(\vec{R}\cdot\vec{\xi})^2 + (\vec{I}\cdot\vec{\xi})^2} + (\vec{R}\cdot\vec{\xi})\right]^{1/2}& \\ + i\sigma_I^{(n+1)} \frac{1}{2\sqrt{2}}\left[+\sqrt{(\vec{R}\cdot\vec{\xi})^2 + (\vec{I}\cdot\vec{\xi})^2} - (\vec{R}\cdot\vec{\xi})\right]^{1/2},&\end{aligned} \tag{78}$$

where σ_R and σ_I are the signatures of the branches of the square root function.

$$\sigma_R^{(n)} = \frac{\mathrm{Re}\,\mathcal{E}_n - \mathrm{Re}\,\mathcal{E}_{n+1}}{|\mathrm{Re}\,\mathcal{E}_n - \mathrm{Re}\,\mathcal{E}_{n+1}|}, \tag{79}$$

$$\sigma_I^{(n)} = \frac{\mathrm{Im}\,\mathcal{E}_n - \mathrm{Im}\,\mathcal{E}_{n+1}}{|\mathrm{Im}\,\mathcal{E}_n - \mathrm{Im}\,\mathcal{E}_{n+1}|} \tag{80}$$

and

$$\sigma_R^{(n+1)} = -\sigma_R^{(n)} \quad \text{and} \quad \sigma_I^{(n+1)} = -\sigma_I^{(n)}. \tag{81}$$

Therefore, when the point (ξ_1, ξ_2) moves on a continuous path in parameter space, the resonance energy eigenvalues (poles of the S-matrix) move in two different continuous trajectories in the unphysical sheet of the complex energy plane. When the point (ξ_1, ξ_2) goes around the exceptional point once in a closed path, it crosses the two branch cut lines once each one, and the two resonance energy eigenvalues, $\mathcal{E}_n(\xi_1, \xi_2)$ and $\mathcal{E}_{n+1}(\xi_1, \xi_2)$, move along two different, non-crossing, continuous trajectories in the complex energy plane in such a way that the $\mathcal{E}_n$ trajectory ends at the point where the $\mathcal{E}_{n+1}$ trajectory starts out, and the $\mathcal{E}_{n+1}$-trajectory ends at the starting point of the $\mathcal{E}_n$-trajectory.

Therefore, when the system goes around the exceptional point once in parameter space, the positions of the resonance energy eigenvalues (poles of the S-matrix) are exchanged in the complex energy plane. Two circuits around the exceptional point in parameter space are required to make the energy eigenvalues return to their initial positions in the complex energy plane.

Now, we may show that the family of functions $\hat{\mathcal{E}}_n(\xi_1, \xi_2)$ and $\hat{\mathcal{E}}_{n+1}(\xi_1, \xi_2)$, given in Eqs. (77) and (78) is a universal unfolding of the degeneracy or crossing point of the two unbound state energy eigenvalues (two resonance poles of the $S(E)$ matrix) in parameter space.

In order to do this, let us introduce a function $\hat{f}_{\text{doub}}(-k; \xi_1, \xi_2)$ related to the contact equivalent approximate pole position functions $\hat{k}_{n,n+1}(\xi_1, \xi_2)$ and $\hat{\mathcal{E}}_{n,n+1}(\xi_1, \xi_2)$, in the same way as $\bar{f}_{\text{doub}}(-k; \xi_1, \xi_2)$ is related to the exact pole position functions $\mathcal{E}_n(\xi_1, \xi_2)$ and $\mathcal{E}_{n+1}(\xi_1, \xi_2)$.

When the regular functions $1/2(k_n(x_1,\ x_2) + k_{n+1}(x_1, x_2))$ and $1/4(k_n(x_1,\ x_2) - k_{n+1}(x_1, x_2))^2$ occurring in Eq. (10) are expanded in a Taylor series about the exceptional point and we keep only the terms of first order, as in Eqs. (31) and (32), we get

$$\bar{f}_{\text{doub}}(-k; \xi_1, \xi_2) = \hat{f}_{\text{doub}}(-k; \xi_1, \xi_2) + \delta \hat{f}_{\text{doub}}(\xi_1, \xi_2), \tag{82}$$

where

$$\hat{f}_{\text{doub}}(-k; \xi_1, \xi_2) = \left[k - \left(k_d + \Delta^{(1)} k_d(x_1, x_2)\right)\right]^2 - \frac{1}{4}((\vec{\mathcal{R}} \cdot \vec{\xi}) + i(\vec{\mathcal{I}} \cdot \vec{\xi})), \tag{83}$$

and

$$\Delta^{(1)} k_d(x_1, x_2) = \sum_{i=1}^{2} d_i^{(1)} \xi_i. \tag{84}$$

The complex constants $d_i^{(1)}$ are the coefficients of the terms of first order in the Taylor series expansion of the function $1/2\,(k_n(\xi_1, \xi_2) + k_{n+1}(\xi_1, \xi_2))$, given in Eqs. (30) and (31), and the components of the real vectors $\vec{\mathcal{R}}$ and $\vec{\mathcal{I}}$ are the real and imaginary parts of the coefficients of the first order terms in the Taylor series expansion of the function $1/4\,(k_n(\xi_1, \xi_2) - k_{n+1}(\xi_1, \xi_2))^2$, given in Eqs. (24) and (30), respectively.

The term $\delta \hat{f}_{\text{doub}}(\xi_1, \xi_2)$ is independent of k, and, as function of (ξ_1, ξ_2) vanishes at the exceptional point as, or faster than ξ_i^2.

From the definition of $\hat{f}_{\text{doub}}(-k;\xi_1,\xi_2)$, Eq. (82), the two functions, $\bar{f}_{\text{doub}}(-k;\xi_1,\xi_2)$ and $\hat{f}_{\text{doub}}(-k;\xi_1,\xi_2)$ are related by

$$\hat{f}_{\text{doub}}(-k;\xi_1,\xi_2)=\zeta(k-d_d;\xi_1,\xi_2)\bar{f}_{\text{doub}}(-k;\xi_1,\xi_2), \tag{85}$$

where $\zeta(k-k_d;\xi_1,\xi_2)$ is a scaling factor

$$\zeta(k-k_d;\xi_1,\xi_2)=1-\frac{\delta\hat{f}_{\text{doub}}(\xi_1,\xi_2)}{\hat{f}_{\text{doub}}(-k;\xi_1,\xi_2)+\delta\hat{f}_{\text{doub}}(\xi_1,\xi_2)} \tag{86}$$

at the exceptional point

$$\zeta(0;0,0)=1. \tag{87}$$

Hence, the function $\bar{f}_{\text{doub}}(-k;\xi_1,\xi_2)$ and $\hat{f}_{\text{doub}}(-k;\xi_1,\xi_2)$ are contact equivalent at the exceptional point (Seydel, 1991).

From the transitivity of the contact equivalence relation, if $\hat{f}_{\text{doub}}(-k;\xi_1,\xi_2)$ and $\bar{f}_{\text{doub}}(-k;\xi_1,\xi_2)$ are contact equivalent at the exceptional point and $\bar{f}_{\text{doub}}(-k;\xi_1,\xi_2)$ and the Jost function $f(-k;\xi_1,\xi_2)$ are also contact equivalent at the exceptional point, it follows that $\hat{f}_{\text{doub}}(-k;\xi_1,\xi_2)$ and the Jost function $f(-k;\xi_1,\xi_2)$ are contact equivalent at the exceptional point.

Then, locally $\hat{f}_{\text{doub}}(-k;\xi_1,\xi_2)$ is contact equivalent to the Jost function $f(-k;\xi_1,\xi_2)$, at the exceptional point. It is also an unfolding (Seydel, 1991; Poston and Stewart, 1978) of $f(-k;\xi_1,\xi_2)$ with the following two features:

1. It includes all possible small perturbation of the degeneracy conditions

$$f(-k;\xi_1,\xi_2)=0, \tag{88}$$

$$\left(\frac{\partial f(-k;\xi_1,\xi_2)}{\partial k}\right)_{k_d}=0, \tag{89}$$

$$\left(\frac{\partial^2 f(-k;\xi_1,\xi_2)}{\partial k^2}\right)_{k_d}\neq 0 \tag{90}$$

up to contact equivalence.

2. It uses the minimum number of parameters, namely two, which is the codimension of the degeneracy (Seydel, 1991; Poston and Stewart, 1978). Here, the two parameters are ξ_1 and ξ_2.

Therefore, $\hat{f}_{\text{doub}}(-k;\xi_1,\xi_2)$ is a universal unfolding (Seydel, 1991) of the Jost function $f(-k;\xi_1,\xi_2)$ at the exceptional point, $\xi_1=0$, $\xi_2=0$ (where $k_{n+1}(0,0)=k_{n,n+1}(0,0)=k_d$).

The vanishing of $\hat{f}_{\text{doub}}(-k;\xi_1,\xi_2)$ defines the approximate wave number-pole position function

$$\hat{k}_{n,n+1}(\xi_1,\xi_2)=k_d+\Delta^{(1)}_{n,n+1}(\xi_1,\xi_2)\pm\left[\frac{1}{4}(\vec{\mathcal{R}}\cdot\vec{\xi}+i\vec{\mathcal{I}}\cdot\vec{\xi})\right]^{1/2} \tag{91}$$

and the corresponding energy-pole position functions $\hat{\mathcal{E}}_{n,n+1}(\xi_1, \xi_2)$ given in Eq. (65)

Since the functions $\hat{\mathcal{E}}_n(\xi_1, \xi_2)$ and $\hat{\mathcal{E}}_{n+1}(\xi_1, \xi_2)$ are obtained from the vanishing of the universal unfolding $\hat{f}_{\text{doub}}(-k; \xi_1, \xi_2)$ of the Jost function $f(-k; \xi_1, \xi_2)$ at the exceptional point, we are justified in saying that, the family of functions $\hat{\mathcal{E}}_n(\xi_1, \xi_2)$ and $\hat{\mathcal{E}}_{n+1}(\xi_1, \xi_2)$, given in Eqs. (77) and (78), is a universal unfolding or deformation of any degeneracy or crossing point of two unbound state energy eigenvalues, which is contact equivalent to the exact energy-pole position function of the isolated doublet of resonances at the exceptional point, and includes all small perturbations of the degeneracy conditions up to contact equivalence.

4. CROSSINGS AND ANTICROSSINGS

The energy-pole position function $\mathcal{E}_{n,n+1}(\xi_1, \xi_2)$ defined in Eqs. (35)–(40) and (48)–(49), may be represented as a hypersurface in a four-dimensional Euclidean space, $\mathcal{E}_4$, with Cartesian coordinates (Re $\epsilon_{n,n+1}$, Im $\epsilon_{n,n+1}, \xi_1, \xi_2$). The space $\mathcal{E}_4$ is the Cartesian product $\epsilon \times \mathbf{P}$ of the complex energy plane ϵ and the (ξ_1, ξ_2)-plane which is the parameter space $\mathbf{P}$ of the physical system. We fix the origin of coordinates in the complex ϵ-plane at the branch point of the pole position function, and the origin of coordinates in parameter space $\mathbf{P}$ at the exceptional point.

4.1. Energy Surfaces

From Eqs. (55)–(58), it can be seen that close to the exceptional point (0, 0), where the two resonance energy eigenvalues become degenerate, the function Re $\hat{\epsilon}_{n,n+1}(\xi_1, \xi_2)$ has two branches and the surface S_R representing this function has two sheets which are glued together from two copies of the plane (ξ_1, ξ_2) which are cut and joined smoothly along the line $\mathcal{L}_R$. The projection of $\mathcal{L}_R$ on the plane (ξ_1, ξ_2) is a line $\mathcal{L}'_R$. The branch cut line $\mathcal{L}_R$ starts at the crossing or critical point, with coordinates (0, 0, 0, 0), and extends from this point into the subspace (0, Im $\hat{\epsilon}_{n,n+1}, \xi_1, \xi_2$) in the *positive* direction defined by the unit vector $\hat{\xi}_o$ satisfying Eq. (58).

The function Im $\hat{\epsilon}_{n,n+1}(\xi_1, \xi_2)$ also has two branches, and the surface S_I representing this function has two sheets which are glued together from two copies of the plane (ξ_1, ξ_2) which are cut and joined smoothly along a line $\mathcal{L}_I$. The projection of the line $\mathcal{L}_I$ on the plane (ξ_1, ξ_2) is also a line $\mathcal{L}'_I$ on the plane (ξ_1, ξ_2). As in the case of Re $\hat{\epsilon}_{n,n+1}(\xi_1, \xi_2)$, the cut starts at the degeneracy (crossing or critical) point with coordinates (0, 0, 0, 0), but in this case the cut extends into the subspace (Re $\hat{\epsilon}_{n,n+1}, 0, \xi_1, \xi_2$) in the *negative* direction defined by the unit vector $\hat{\xi}_c$ satisfying Eq. (58).

The lines $\mathcal{L}_R$ and $\mathcal{L}_I$ are in orthogonal subspaces, but have one point in common, the exceptional point $(0, 0) \epsilon$ **P**.

The projections of the lines $\mathcal{L}_R$ and $\mathcal{L}_I$, on the plane (ξ_1, ξ_2) are the two halves of the line $\mathcal{L}'$. Both halves of $\mathcal{L}'$ start at the exceptional point, but they extend in opposite directions.

4.2. Sections of the Energy Surfaces

Let us consider a point (ξ_1, ξ_2) in parameter space away from the exceptional point. To this point corresponds a pair of non-degenerate resonance energy eigenvalues

$$\mathcal{E}_n(\xi_1, \xi_2) \neq \mathcal{E}_{n+1}(\xi_1, \xi_2), \quad (\xi_1, \xi_2) \neq (0, 0) \epsilon \mathbf{P}. \tag{92}$$

These two resonance energy eigenvalues are represented by two points on the hypersurface representing the function $\epsilon_{n,n+1}(\xi_1, \xi_2)$ defined in Eq. (45).

When the point (ξ_1, ξ_2) traces a path π in parameter space, the corresponding points $\mathcal{E}_n(\xi_1, \xi_2)$ and $\mathcal{E}_{n+1}(\xi_1, \xi_2)$ trace two curving trajectories, $C_n(\pi)$ and $C_{n+1}(\pi)$ on the $\epsilon_{n,n+1}(\xi_1, \xi_2)$ hypersurface.

The contact equivalent approximant $\hat{\epsilon}_{n,n+1}(\xi_1, \xi_2)$ of $\epsilon_{n,n+1}(\xi_1, \xi_2)$ may also be represented as a hypersurface in $\mathcal{E}_4$. In this case, to a point $(\xi_1, \xi_2) \neq (0, 0)$ in parameter space corresponds a pair of points

$$\hat{\mathcal{E}}_n(\xi_1, \xi_2) \neq \hat{\mathcal{E}}_{n+1}(\xi_1, \xi_2), \quad (\xi_1, \xi_2) \neq (0, 0) \ \epsilon \ \mathcal{M} \subset \mathbf{P} \tag{93}$$

on the $\hat{\epsilon}_{n,n+1}(\xi_1, \xi_2)$-hypersurface. When the point (ξ_1, ξ_2) traces a path π in parameter space, the corresponding points $\hat{\mathcal{E}}_n(\xi_1, \xi_2)$ and $\hat{\mathcal{E}}_{n+1}(\xi_1, \xi_2)$ trace two curving trajectories, $\hat{C}_n(\pi)$ and $\hat{C}_{n+1}(\pi)$ on the $\hat{\epsilon}_{n,n+1}(\xi_1, \xi_2)$-hypersurface.

The topological structure of the hypersurfaces $\epsilon_{n,n+1}(\xi_1, \xi_2)$ and $\hat{\epsilon}_{n,n+1}(\xi_1, \xi_2)$ close to the crossing point will be most clearly evident in the shape and properties of the trajectories, $C_n(\pi)$ and $C_{n+1}(\pi)$ and, $\hat{C}_n(\pi)$ and $\hat{C}_{n+1}(\pi)$, respectively, for paths that cross the line $\mathcal{L}'$ at points close to the exceptional point.

We define three straight line paths in parameter space, π_1, π_2 and π_3, by keeping the parameter ξ_2 fixed at some value $\bar{\xi}_2^{(i)}$

$$\xi_2 = \bar{\xi}_2^{(i)}, \quad i = 1, 2, 3, \tag{94}$$

and letting ξ_1 vary. The values of $\bar{\xi}_2^{(i)}$ are chosen in such a way that the paths, π_1, π_2 and π_3, cross the line $\mathcal{L}'$ at points located just before, at, and just after the exceptional point. That is, π_1 crosses the line $\mathcal{L}'_I$ at a point just before the exceptional point, π_2 crosses the line $\mathcal{L}'$ at the exceptional point which is the only point that $\mathcal{L}'_R$ and $\mathcal{L}'_I$ have in common, π_3 crosses the line $\mathcal{L}'_R$ at a point just after the exceptional point.

The condition (94) also defines three hyperplanes in the space $\mathcal{E}_4$. The intersections of the hypersurface $\epsilon_{n,n+1}(\xi_1, \xi_2)$ with the hyperplanes, $\xi_2 = \bar{\xi}_2^{(i)}$, are the trajectories $C_n(\pi_i)$ and $C_{n+1}(\pi_i)$ and the intersection of the hypersurface $\hat{\epsilon}_{n,n+1}(\xi_1, \xi_2)$ with the hyperplanes $\xi_2 = \bar{\xi}_2^{(i)}$ are the trajectories $\hat{C}(\pi_i)$ and $\hat{C}_{n+1}(\pi_i)$.

In the general case, the energy-pole position function, $\mathcal{E}_{n,n+1}(\xi_1, \xi_2)$, is not known as an explicit function of the control parameters of the system. The trajectories, $C_n(\pi_i)$ and $C_{n+1}(\pi_i)$, may be computed numerically only after the equation defining implicitly the pole position function is solved numerically. In the general case, this is a rather arduous task.

In a companion paper (Hernández *et al.*, 2007), we give the results of solving numerically the implicit Eqs. (16)–(18) for the pole position function, $k_{n,n+1}(\xi_1, \xi_2)$, in the case of a degeneracy of an isolated doublet of resonances in the scattering of a beam of particles by a double barrier potential well with two regions of trapping. The resulting Re $k_{n,n+1}(\xi_1, \xi_2)$, Im $k_{n,n+1}(\xi_1, \xi_2)$ and the trajectories $C_n(\pi_i)$ and $C_{n+1}(\pi_i)$ are shown as three dimensional graphs Figs. 8, 9 and 10, of that paper (Hernández *et al.*, 2007).

When the functional dependence of the Jost function on the control parameters is known and the coefficients $c_i^{(1)}$ and $C_i^{(1)}$ occurring in Eqs. (34) and (40) may be calculated, the contact equivalent approximant for the pole position function, $\hat{\mathcal{E}}_{n,n+1}(\xi_1, \xi_2)$, and the contact equivalent approximant trajectories, $\hat{C}_n(\pi_i)$, and $\hat{C}_{n+1}(\pi_i)$ may be analytically computed from Eqs. (40), (55)–(57) and (77) and (78), as will be explained below. In Figs. 1, 2 and 3, in this paper we show the trajectories $\hat{C}_n(\pi_i)$ and $\hat{C}_{n+1}(\pi_i)$ for the example discussed in Hernández *et al.* (2007). In these figures, the numerically computed exact trajectories $C_n(\pi_i)$ and $C_{n,n+1}(\pi_i)$ are not shown, because they can hardly be distinguished from the contact aproximants $\hat{C}_n(\pi_i)$ and $\hat{C}_{n+1}(\pi_i)$.

4.3. Projections

The trajectories, $\hat{C}_n(\pi_i)$ and $\hat{C}_{n+1}(\pi_i)$, are the intersection of the surface $\hat{\epsilon}_{n,n+1}$ and the hyperplanes $\xi_2 = \bar{\xi}_2^{(i)}$ in the space $\mathcal{E}_4$ with Cartesian coordinates (Re $\mathcal{E}$, Im $\mathcal{E}$, ξ_1, ξ_2). Since ξ_2 is kept constant at the fixed value $\bar{\xi}_2^{(i)}$, the trajectories (sections), $\hat{C}_n(\pi_i)$ and $\hat{C}_{n+1}(\pi_i)$, may be represented as three dimensional curves in a space $\mathcal{E}_3$ with Cartesian coordinates (Re $\mathcal{E}$, Im $\mathcal{E}$, ξ_1).

The projections of the trajectories $\hat{C}_n(\pi_i)$ and $\hat{C}_{n+1}(\pi_i)$, on the plane (Re $\mathcal{E}$, ξ_1) are

$$\text{Re}\,[\hat{C}_n(\pi_i)] = \text{Re}\,\hat{\mathcal{E}}_n\big(\xi_1, \hat{\xi}_2^{(i)}\big) \tag{95}$$

and

$$\text{Re}\,[\hat{C}_{n+1}(\pi_i)] = \text{Re}\,\hat{\mathcal{E}}_{n+1}\big(\xi_1, \hat{\xi}_2^{(i)}\big). \tag{96}$$

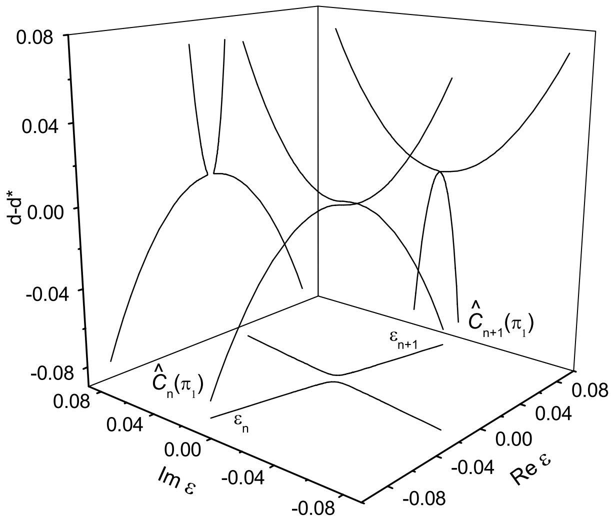

Fig. 1. The curves $\hat{C}_n(\pi_1)$ and $\hat{C}_{n+1}(\pi_1)$ are the trajectories traced by the points $\hat{\mathcal{E}}_n(d, V_3)$ and $\hat{\mathcal{E}}_{n+1}(d, V_3)$ on the hypersurface $\hat{\mathcal{E}}_{n,n+1}(d, V_3)$ when the point (d, V_3) moves along the straight line path π_1 in parameter space. In the figure, the path π_1 runs parallel to the vertical axis and crosses the line $\mathcal{L}_I$ at a point $(d^{(1)}, V_3^{(1)})$ with $d^{(1)} < d^*$ and $\bar{V}_3^{(1)} < V_3^*$. The projections of $\hat{C}_n(\pi_1)$ and $\hat{C}_{n+1}(\pi_1)$ on the plane $(\text{Im } \mathcal{E}, d)$ are sections of the surface S_I; the projections of $\hat{C}_n(\pi_1)$ and $\hat{C}_{n+1}(\pi_1)$ on the plane $(\text{Re } \mathcal{E}, d)$ are sections of the surface S_R. The projections of $\hat{C}_n(\pi_1)$ and $\hat{C}_{n+1}(\pi_1)$ on the plane $(\text{Re } \mathcal{E}, \text{Im } \mathcal{E})$ are the trajectories of the S-matrix poles in the complex energy plane.

Analytical expressions for the right hand sides of these equations are obtained setting $\xi_2 = \bar{\xi}_2^{(i)}$ in Eqs. (77) and (78). In this way, we get

$$\text{Re } [\hat{\mathcal{E}}_n(\xi_1, \bar{\xi}_2^{(i)})] = \frac{\sigma_R^{(n)}}{2\sqrt{2}} |c_1^{(1)}| [_+\sqrt{\xi_1^2 + 2z_i \cos(\phi_1 - \phi_2)\xi_1 + z_i^2} + (\cos\phi_1 \xi_1 + \cos\phi_2 z_i)]^{1/2}. \tag{97}$$

The constants that appear in the right hand side of this equation are obtained from the expansion coefficients $C_j^{(1)}$, $\vec{R}$ and $\vec{I}$ occurring in Eqs. (47) and (55)–(57)

$$z_i = \left| \frac{c_2^{(1)}}{c_1^{(1)}} \right| \bar{\xi}_2^{(i)}, \tag{98}$$

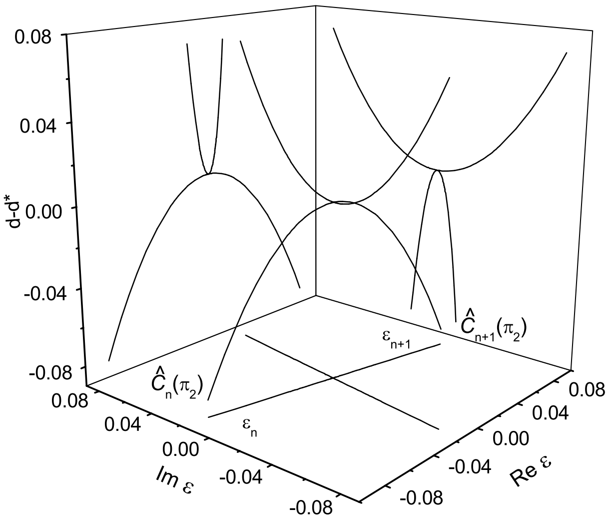

Fig. 2. The curves $\hat{C}_n(\pi_2)$ and $\hat{C}_{n+1}(\pi_2)$ are the trajectories of the points $\hat{\mathcal{E}}_n(d, V_3^*)$ and $\hat{\mathcal{E}}_{n+1}(d, V_3^*)$ on the hypersurface $\hat{\mathcal{E}}_{n,n+1}(d, V_3)$ when the point (d, V_3^*) moves along a straight line path π_2 that goes through the exceptional point (d^*, V_3^*) in parameter space. The projections of $\hat{C}_n(\pi_2)$ and $\hat{C}_{n+1}(\pi_2)$ on the planes (Re $\mathcal{E}$, d) and (Im $\mathcal{E}$, d) are sections of the surfaces S_R and S_I, respectively, and show a joint crossing of energies and widths. The projections of $\hat{C}_n(\pi_2)$ and $\hat{C}_{n+1}(\pi_2)$ on the plane (Re $\mathcal{E}$, Im $\mathcal{E}$) are two straight line trajectories of the S-matrix poles – crossing at 90° in the complex energy plane. At the crossing point, the two simple poles coalesce into one double pole of $S(E)$.

and

$$\cos\phi_1 = \frac{R_1}{\left|c_1^{(1)}\right|}, \quad \cos\phi_2 = \frac{R_2}{\left|c_2^{(1)}\right|}. \tag{99}$$

From (78) and (81), we get

$$\text{Re}\,[\hat{C}_{n+1}(\pi_i)] = -\text{Re}\,[\hat{C}_n(\pi_i)]. \tag{100}$$

The projections of the trajectories $\hat{C}_n(\pi)$ and $\hat{C}_{n+1}(\pi_i)$ on the plane (Im $\mathcal{E}$, ξ_1) are obtained from a similar argument

$$\text{Im}\,[\hat{C}_n(\pi_i)] = \text{Im}\,\hat{\mathcal{E}}_n\big(\xi_1, \bar{\xi}_2^{(i)}\big), \tag{101}$$

where

$$\text{Im}\,\hat{\mathcal{E}}_n\big(\xi_1, \bar{\xi}_2^{(i)}\big) = \frac{\sigma_I^{(n)}}{2\sqrt{2}}\left|c_1^{(1)}\right|\Big[{}_{+}\sqrt{\xi_1^2 + 2z_i\cos(\phi_1 - \phi_2)\xi_i + z_i^2} \\ -(\cos\phi_1\xi_1 + \cos\phi_2 z_1)\Big]^{1/2} \tag{102}$$

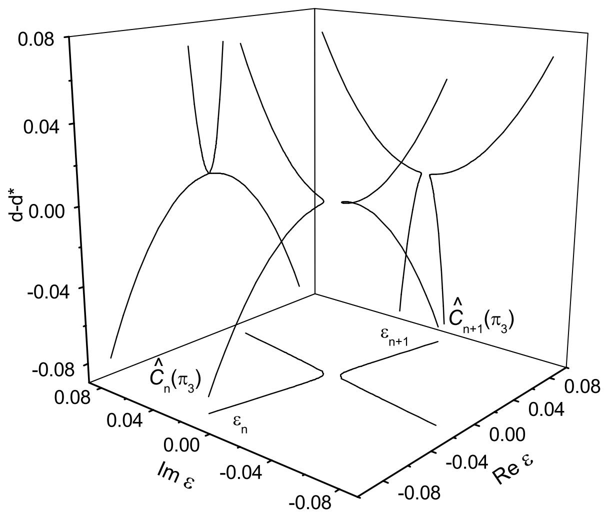

Fig. 3. The curves $\hat{C}_n(\pi_3)$ and $\hat{C}_{n+1}(\pi_3)$ are the trajectories traced by the points $\hat{\mathcal{E}}_n(d, \bar{V}_3^{(3)})$ and $\hat{\mathcal{E}}_{n+1}(d, \bar{V}_3^{(3)})$ on the hypersurface $\mathcal{E}_{n,n+1}(d, V_3)$ when the point $(d, \bar{V}_3^{(3)})$ moves along a straight line path π_3 going through the point $(\bar{d}^{(3)}, \bar{V}_3^{(3)})$ with $\bar{d}^{(3)} > d^*$. The path π_3 crosses the line $\mathcal{L}_R$. The projections of $\hat{C}_n(\pi_3)$ and $\hat{C}_{n+1}(\pi_3)$ on the plane (Re $\mathcal{E}$, d) show a crossing, but the projections on the planes (Im $\mathcal{E}$, d) and (Re $\mathcal{E}$, Im $\mathcal{E}$) do not cross.

and

$$\text{Im}\,[\hat{C}_{n+1}(\pi_i)] = -\text{Im}\,[\hat{C}_n(\pi_i)]. \tag{103}$$

4.4. Crossings and Anticrossings of Energies and Widths

The curves representing Re $[\hat{C}_n(\pi_i)]$ and Re $[\hat{C}_{n+1}(\pi_i)]$ cross in the plane (Re $\epsilon_{n,n+1}$, ξ_1), see Fig. 3, at a point $\bar{\xi}_c = (\xi_{1c}, \bar{\xi}_2^{(i)})$, where

$$\text{Re}\,[\hat{C}_n(\pi_i)]|_{\vec{\xi}=\vec{\xi}_c} = \text{Re}\,[\hat{C}_{n+1}(\pi_i)]|_{\vec{\xi}=\vec{\xi}_c}. \tag{104}$$

From Eqs. (100), (77) and (78), this condition means that the crossing occurs only if

$$\left[+\sqrt{(\vec{R}\cdot\vec{\xi}_c)^2 + (\vec{I}\cdot\vec{\xi}_c)^2} + \vec{R}\cdot\vec{\xi}_c\right]\Big|_{\xi_2=\bar{\xi}_2^{(i)}} = 0, \tag{105}$$

but this condition is satisfied only for points $\vec{\xi}_c$ on the branch cut line $\mathcal{L}_R$.

Therefore, the crossing point of the curves representing Re $[\hat{C}_n(\pi_i)]$ and Re $[\hat{C}_{n+1}(\pi_i)]$ is the intersection of the hyperplane $\xi_2 = \bar{\xi}_2^{(i)}$ and the branch cut line $\mathcal{L}_R$.

From Eqs. (103), (77) and (78), a similar argument shows that the curves Im $[\hat{C}_n(\pi_i)]$ and Im $[\hat{C}_{n+1}(\pi_i)]$ cross in the plane (Im $\epsilon_{n,n+1}, \xi_1$), at a point $\vec{\xi'}_c = (\xi'_{1c}, \bar{\xi}_2^{(i)})$ where,

$$\text{Im}\,[\hat{C}_n(\pi_i)]|_{\vec{\xi'}=\vec{\xi'}_c} = \text{Im}\,[\hat{C}_{n+1}(\pi_i)]|_{\vec{\xi'}=\vec{\xi'}_c}, \tag{106}$$

but, from (77) and (78), this condition means that

$$\left[+\sqrt{(\vec{R}\cdot\vec{\xi'}_c)^2 + (\vec{I}\cdot\vec{\xi'}_c)^2} - \vec{R}\cdot\vec{\xi'}_c\right]|_{\xi'_2=\bar{\xi}_2^{(i)}} = 0, \tag{107}$$

but this is the condition that defines the branch cut $\mathcal{L}_I$.

Hence, the curves representing Im $[\hat{C}_n(\pi_j)]$ and Im $[\hat{C}_{n+1}(\pi_j)]$ cross in the plane (Im $\epsilon_{n,n+1}, \xi_1$) at a point (Im $\epsilon_{n,n+1} = 0, \xi_1 = \xi'_{1c}$) only when the hyperplane $\xi_2 = \bar{\xi}_2^{(i)}$ intersects the branch cut line $\mathcal{L}_I$ at the point $\vec{\xi'}_c = (\xi'_{1c}, \bar{\xi}_2^{(j)})$, see Fig. 1.

The branch cut lines, $\mathcal{L}_R$ and $\mathcal{L}_I$, have one point in common, namely, the exceptional point. Hence, when the path π_2 crosses the line $\mathcal{L}'$ exactly at the exceptional point where $\mathcal{L}_R$ and $\mathcal{L}_I$ meet, both the real and imaginary parts of the trajectories (sections) $\hat{C}_n(\pi_2)$ and $\hat{C}_{n+1}(\pi_2)$ cross, and the two complex energy eigenvalues are equal, that is, at the point of exact degeneracy of resonances, see Fig. 2.

The interpretation of the trajectories $\hat{C}_n(\pi)$ and $\hat{C}_{n+1}(\pi)$ as sections of the energy sheets representing the complex energy eigenvalues $\hat{\mathcal{E}}_n(\xi_1, \xi_2)$ and $\hat{\mathcal{E}}_{n+1}(\xi_1, \xi_2)$, leads in a natural way to the interpretation of the crossing and anticrossing properties of the projections Re $\hat{\mathcal{E}}_n(\xi_1, \xi_2)$, Re $\hat{\mathcal{E}}_{n+1}(\xi_1, \xi_2)$ and Im $\hat{\mathcal{E}}_n(\xi_1, \xi_2)$, Im $\hat{\mathcal{E}}_{n+1}(\xi_1, \xi_2)$ in terms of the topological properties of the energy hypersurfaces close to the degeneracy point. Now, it will be convenient to rephrase this interpretation in more physical terms.

In standard notation, Re $\mathcal{E}_n(\xi_1, \xi_2)$ is the resonance energy E_n and– Im $\mathcal{E}_n(\xi_1, \xi_2)$ is the resonance half-width $1/2(\Gamma_n)$, that is

$$\mathcal{E}_n(\xi_1, \xi_2) = E_n - i\frac{1}{2}\Gamma_n \tag{108}$$

and a similar expression for $\mathcal{E}_{n+1}(\xi_1, \xi_2)$.

When the physical system is perturbed by allowing one of the external parameters to vary, say ξ_1, while the other parameter is kept constant, the energies and widths of the resonances change and the two unbound states in the isolated doublet of resonances get mixed.

In order to describe the mixing of the two unbound states, it is useful to consider the differences of the energies ΔE, and the differences of the widths $\Delta\Gamma$ of the two perturbed states.

From Eqs. (77)–(81), and keeping $\xi_2 = \bar{\xi}_2^{(i)}$, we obtain

$$\Delta E = E_n - E_{n+1} = (\text{Re}\,\hat{\mathcal{E}}_n - \text{Re}\,\hat{\mathcal{E}}_{n+1})\big|_{\xi_2=\bar{\xi}_2^{(i)}}$$

$$= \frac{\sigma_R^{(n)}\sqrt{2}}{2}\left[+\sqrt{(\vec{R}\cdot\vec{\xi})^2 + (\vec{I}\cdot\vec{\xi})^2} + (\vec{R}\cdot\vec{\xi})\right]^{1/2}\big|_{\xi_2=\bar{\xi}_2^{(i)}} \tag{109}$$

and

$$\Delta\Gamma = \frac{1}{2}(\Gamma_n - \Gamma_{n+1}) = \text{Im}\,(\mathcal{E}_{n+1}) - (\text{Im}\,\mathcal{E}_n)$$

$$= -\frac{\sigma_I^{(n)}\sqrt{2}}{2}\left[+\sqrt{(\vec{R}\cdot\vec{\xi})^2 + (\vec{I}\cdot\vec{\xi})^2} - (\vec{R}\cdot\vec{\xi})\right]^{1/2}\big|_{\xi_2=\bar{\xi}_2^{(i)}}. \tag{110}$$

These expressions allow us to relate the terms $(\vec{R}\cdot\vec{\xi})$ and $(\vec{I}\cdot\vec{\xi})$ directly with observables of the isolated doublet of resonances.

Taking the product of $\Delta E\Delta\Gamma$, and recalling Eq. (57), we get

$$\Delta E\Delta\Gamma = -\frac{1}{2}(\vec{I}\cdot\vec{\xi})|_{\xi_2=\bar{\xi}_2^{(i)}} \tag{111}$$

and taking the differences of the squares of the left hand sides of (109) and (110), we get

$$(\Delta E)^2 - (\Delta\Gamma)^2 = (\vec{R}\cdot\vec{\xi})|_{\xi_2=\bar{\xi}_2^{(i)}}. \tag{112}$$

At a crossing of energies ΔE vanishes and at a crossing of widths $\Delta\Gamma$ vanishes. Hence, the relation found in Eq. (111) means that a crossing of energies or widths can occur if and only if $(\vec{I}\cdot\vec{\xi})_{\bar{\xi}_2^{(i)}}$ vanishes.

For a vanishing $(\vec{I}\cdot\vec{\xi}_c)_{\bar{\xi}_2^{(i)}} = 0 = \Delta E\Delta\Gamma$, we find three cases, which are distinguished by the sign of $(\vec{R}\cdot\vec{\xi}_c)_{\bar{\xi}_2^{(i)}}$. From Eqs. (109) and (110),

1. $(\vec{R}\cdot\vec{\xi}_c)_{\bar{\xi}_2^{(i)}} > 0$ implies $\Delta E \neq 0$ and $\Delta\Gamma = 0$, i.e. energy anticrossing and width crossing.
2. $(\vec{R}\cdot\vec{\xi}_c)_{\bar{\xi}_2^{(i)}} = 0$ implies $\Delta E = 0$ and $\Delta\Gamma = 0$, that is, joint energy and width crossing, which is also degeneracy of the two complex resonance energy eigenvalues.
3. $(\vec{R}\cdot\vec{\xi}_c)_{\bar{\xi}_2^{(i)}} < 0$ implies $\Delta E = 0$ and $\Delta\Gamma \neq 0$, i.e. energy crossing and width anticrossing.

This rich physical scenario of crossings and anticrossings for the energies and widths of the complex resonance energy eigenvalues, extends a theorem of von Neumann and Wigner (1929) for bound states to the case of unbound states.

In the case of two bound states, the energy eigenvalues, E_1 and E_2, are real. When the perturbation depends on one external parameter, the difference $\Delta E = E_1 - E_2$, cannot vanish. That is, the energies of two bound states repel (anticross) for any non-vanishing value of a perturbation depending on only one parameter (von Neumann and Wigner, 1929; Berry, 1985).

The general character of the crossing-anticrossing relations of the energies and widths of a mixing isolated doublet of resonances, discussed above, has been experimentally established by P. von Brentano and his collaborators in a series of beautiful experiments (von Brentano and Philipp, 1999; Philipp *et al.*, 2000; von Brentano, 2002).

5. TRAJECTORIES OF THE *S*-MATRIX POLES AND CHANGES OF IDENTITY

The trajectories of the S-matrix poles (complex resonances energy eigenvalues), $\hat{\mathcal{E}}_n(\xi_1, \xi_2)$ and $\hat{\mathcal{E}}_{n+1}(\xi_1, \xi_2)$, in the complex energy plane are the projections of the three-dimensional trajectories (sections) $\hat{C}_n(\pi_i)$ and $\hat{C}_{n+1}(\pi_i)$ on the plane (Re ϵ, Im ϵ), see Figs. 1, 2, and 3.

An equation for the trajectories of the S-matrix poles in the complex energy plane is obtained by eliminating ξ_1 between Re $\hat{\mathcal{E}}_n(\xi_1, \bar{\xi}_2^{(i)})$ and Im $\hat{\mathcal{E}}_n(\xi_1, \bar{\xi}_2^{(i)})$, Eqs. (97) and (102).

A straightforward calculation gives

$$\mathrm{Re}\,(\hat{\mathcal{E}}_n)^2 - 2\cot\phi_1(\mathrm{Re}\,\hat{\mathcal{E}}_n)(\mathrm{Im}\,\hat{\mathcal{E}}_n) - (\mathrm{Im}\,\hat{\mathcal{E}}_n)^2 - \frac{1}{4}\left(\vec{R}\cdot\xi_c^{(i)}\right) = 0, \qquad (113)$$

where

$$\cot\phi_1 = \frac{R_1}{I_1} \qquad (114)$$

and the constant vector $\vec{\xi}_c^{(i)}$ is such that,

$$(\vec{I}\cdot\vec{\xi}_c)|_{\xi_2=\bar{\xi}_2^{(i)}} = 0, \qquad (115)$$

which is the previously found condition for the occurrence of a crossing of ΔE or $\Delta\Gamma$.

The discriminant of Eq. (113), $4(\cot^2\phi_1 + 1)$, is positive. Therefore, close to the crossing point, the trajectories of the S-matrix poles are the branches of a hyperbola defined by Eq. (113).

The asymptotes of the hyperbola are the two straight lines defined by

$$\mathrm{Im}\,\mathcal{E}^{(I)} = \tan\frac{\phi_1}{2}\mathrm{Re}\,\mathcal{E}^{(I)} \qquad (116)$$

and

$$\mathrm{Im}\,\mathcal{E}^{(II)} = -\cot\frac{\phi_1}{2}\mathrm{Re}\,\mathcal{E}^{(II)}. \qquad (117)$$

The two asymptotes divide the complex energy plane in four quadrants. The two branches of the hyperbola are in opposite, that is, not adjacent, quadrants of the complex energy plane.

We verify that, if $\mathcal{E}_n$ satisfies Eq. (113), so does $-\mathcal{E}_n = \mathcal{E}_{n+1}$. Therefore, if the trajectory followed by the pole $\mathcal{E}_n$ is one branch of the hyperbola, the trajectory followed by the pole $\mathcal{E}_{n+1}$ is the other branch of the hyperbola. Initially, the poles move towards each other from opposite ends of the two branches of the hyperbola until they come close to the crossing point, then they move away from each other, each pole on its own branch of the hyperbola.

We find three types of trajectories, which are distinguished by the sign of $(\vec{R} \cdot \vec{\xi}_c)|_{\xi_2=\bar{\xi}_2^{(i)}}$.

1. Trajectories of type I, when $(\vec{R} \cdot \vec{\xi}_c)|_{\xi_2=\bar{\xi}_2^{(i)}} < 0$.

   In this case there is a crossing of energies and an anticrossing of widths. Hence, one branch of the hyperbola, say, the trajectory followed by the pole $\mathcal{E}_n$, lies above a horizontal straight line, parallel to the real axis, and going through the crossing point $\mathcal{E}_d$. The other branch of the hyperbola, the trajectory followed by the pole $\mathcal{E}_{n+1}$, lies below the horizontal line, parallel to the real axis, going through the crossing point $\mathcal{E}_d$, see Fig. 4.
2. Critical trajectories (type II), when $(\vec{R} \cdot \vec{\xi}_c)|_{\xi_2=\bar{\xi}_2^{(i)}} = 0$.

   There is a joint crossing of energies and widths. The trajectories are the asymptotes of the hyperbola.

   The two poles, $\mathcal{E}_n$ and $\mathcal{E}_{n+1}$, start from opposite ends of the same straight line, and move towards each other until they meet at the crossing point, where they coalesce to form a double pole of the S-matrix. From here, they separate moving away from each other on a straight line at 90° with respect to the first asymptote, see Fig. 5.
3. Trajectories of type III, when $(\vec{R} \cdot \vec{\xi}_c)|_{\xi_2=\bar{\xi}_2^{(i)}} > 0$.

   In this case, there is an anticrossing of energies and a crossing of widths. Therefore, one branch of the hyperbola, say, the trajectory followed by the pole $\mathcal{E}_n$, lies to the left of a vertical straight line, parallel to the imaginary axis and going through the crossing point $\mathcal{E}_d$. The other branch of the hyperbola, the trajectory followed by the pole $\mathcal{E}_{n+1}$, lies to the right of the line parallel to the imaginary axis that goes through the crossing point $\mathcal{E}_d$, see Fig. 6.

It is interesting to notice that, a small change in the external control parameter $\bar{\xi}_2^{(i)}$ produces a small change in the initial position of the poles, $\mathcal{E}_n$ and $\mathcal{E}_{n+1}$, but when the small change in $\bar{\xi}_2^{(i)}$ changes the sign of $(\vec{R} \cdot \vec{\xi}_c)|_{\xi_2=\bar{\xi}_2^{(i)}}$, the trajectories change suddenly from type I to type III, this very large and sudden change of the trajectories exchanges almost exactly the final positions of the poles $\mathcal{E}_n$ and $\mathcal{E}_{n+1}$. This dramatic change has been termed a "change of identity" by Vanroose *et al.*

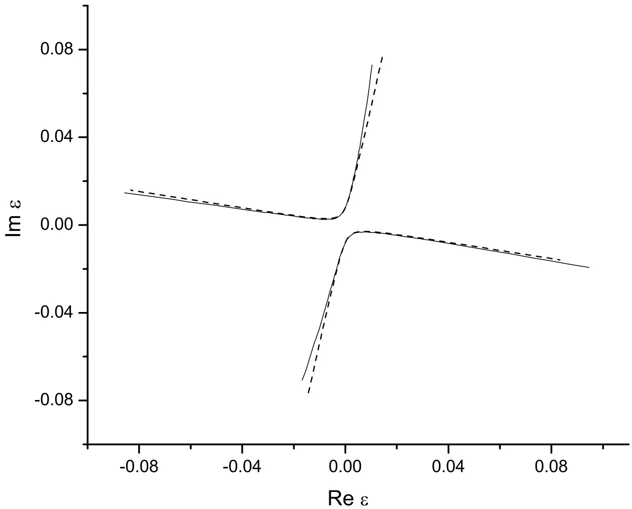

Fig. 4. Trajectories of the $S(E)-$matrix poles of an isolated doublet of resonances in a double barrier potential. The trajectories are traced by the poles $\mathcal{E}_n(d, \bar{V}_3^{(1)})$ and $\mathcal{E}_{n+1}(d, \bar{V}_3^{(1)})$ on the complex energy plane when the point $(d, \bar{V}_3^{(1)})$ moves on the straight line path π_1 that crosses the line $\mathcal{L}'_R$ at $(d^{(1)}, \bar{V}_3^{(1)})$. There is crossing of energies $(\Delta E = 0)$ and an anticrossing of widths $(\Delta\Gamma \neq 0)$. The full lines are the exact trajectories obtained from a numerical computation of the energy pole position function (Hernández *et al.*, 2003a, 2007). The dashed lines are obtained from the contact approximants $\hat{\mathcal{E}}_n(d, V_3)$ and $\hat{\mathcal{E}}_{n+1}(d, V_3)$ described in the text.

(1997) who discussed an example of this phenomenon in the S-matrix poles in a two-channel model, W. Vanroose (2001) and Hernández *et al.* (2003a) have also discussed these properties in the case of the scattering of a beam of particles by a double barrier potential with two regions of trapping.

6. SUMMARY AND CONCLUSIONS

We have investigated the degeneracy of an isolated doublet of resonance energy eigenvalues of a quantum system as functions of the control parameters of the system. The aim was to give a theoretical explanation of the experimentally well established rich physical scenario of crossings and anticrossings of energies and widths of the mixing resonances in the isolated doublet. We were also able to explain the large and sudden change in the shape of the trajectories of the S-matrix poles in the complex energy plane observed when the control parameters suffer a very small change in the vicinity of the exceptional point.

We proceeded in four steps:

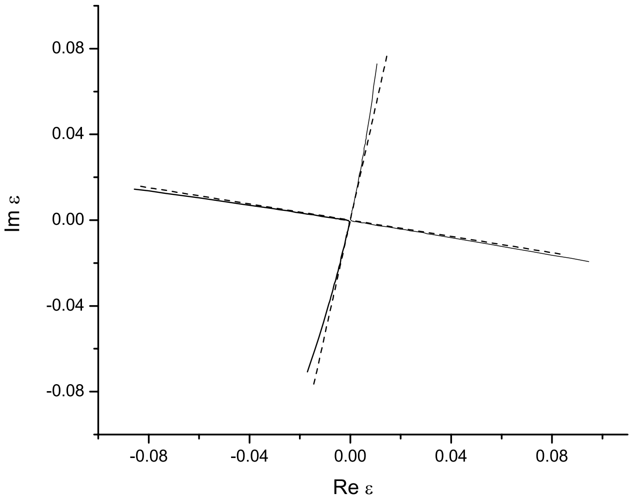

Fig. 5. Trajectories of the $S(E)$-matrix poles of an isolated doublet of unbound states in a double barrier potential. The trajectories are traced by the poles $\mathcal{E}_n(d, V^*)$ and $\mathcal{E}_{n+1}(d, V^*)$ on the complex energy plane when the point (d, V^*) moves on the straight line path π_2 that goes through the exceptional point (d^*, V_3^*). There is a joint crossing of energies $(\Delta E = 0)$ and widths $(\Delta\Gamma = 0)$ at exact degeneracy where the two simple poles meet and coalesce to form one double pole of $S(E)$. The full lines are the exact trajectories obtained from a numerical computation of the energy-pole position function (Hernández *et al.*, 2003a, 2007). The dashed lines were obtained from the contact approximants $\hat{\mathcal{E}}_n(d, V_3)$ and $\hat{\mathcal{E}}_{n+1}(d, V_3)$ described in the text.

1. The degeneracy of the resonance energy eigenvalues was implicitly defined by the vanishing of the Jost function and its first derivative with respect to the wave number k. By formally solving these equations, we introduced an explicit expression for the double valued energy-pole position function $\mathcal{E}_{n,n+1}(\xi_1, \xi_2)$ of the isolated doublet of resonances. The function $\mathcal{E}_{n,n+1}(\xi_1, \xi_2)$ determines the position of the S-matrix poles of the doublet in the complex energy plane, for each set of values of the control parameters (ξ_1, ξ_2). Then we showed that the semi-sum and the square of the semi-difference of the resonance energy eigenvalues, $1/2\,(\mathcal{E}_n(\xi_1, \xi_2) + \mathcal{E}_{n+1}(\xi_1, \xi_2))$ and $1/4\,(\mathcal{E}_n(\xi_1, \xi_2) - \mathcal{E}_{n+1}(\xi_1, \xi_2))^2$, that appear in $\mathcal{E}_{n,n+1}(\xi_1, \xi_2)$ are regular functions of the control parameters at the exceptional point where the degeneracy occurs.
2. From these results, we obtained a contact equivalent approximant $\hat{\mathcal{E}}_{n,n+1}(\xi_1, \xi_2)$ to the pole position function $\mathcal{E}_{n,n+1}(\xi_1, \xi_2)$ which gave us

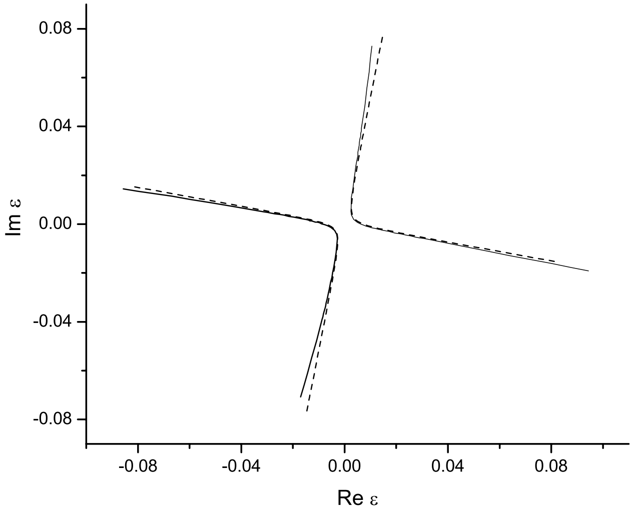

Fig. 6. Trajectories of the $S(E)$-matrix poles of an isolated doublet of resonances in a double barrier potential. The trajectories are traced by the poles $\mathcal{E}_n(d, \bar{V}_3^{(3)})$ and $\mathcal{E}_{n+1}(d, \bar{V}_3^{3)})$ on the complex energy plane when the point $(d, \bar{V}_3^{(3)})$ moves on the straight line π_3 that crosses the line $\mathcal{L}'_I$ at $(d^{(3)}, \bar{V}_3^{(3)})$ in parameter space. There is an anticrossing of energies ($\Delta E \neq 0$) and a crossing of widths ($\Delta\Gamma = 0$). The full lines are the exact trajectories obtained from a numerical computation of the energy-pole position function (Hernández *et al.*, 2003a, 2007). The dashed lines are obtained from the contact approximants $\hat{\mathcal{E}}_n(d, V_3)$ and $\hat{\mathcal{E}}_{n+1}(d, V_3)$ described in the text. When comparing this figure with Fig. 4, notice the "change of identity" brought about by a small change in (d_3, V_3).

a simple and explicit but very accurate representation of the analytical behavior of the pole position function $\mathcal{E}_{n,n+1}(\xi_1, \xi_2)$ as function of the control parameters in the vicinity of the crossing point. We found that, close to the exceptional point:

the real part of the energy-pole position function Re $\mathcal{E}_{n,n+1}(\xi_1, \xi_2)$, as function of the real parameters (ξ_1, ξ_2), has an algebraic branch point of square root type (rank one) at the exceptional point in parameter space, and a branch cut along a line $\mathcal{L}_R$, that starts at the exceptional point and extends in the *positive* direction defined by a vector $\vec{\xi}_c$ satisfying $(\vec{I} \cdot \vec{\xi}_c) = 0$.

The imaginary part of the energy pole position function Im $\mathcal{E}_{n,n+1}(\xi_1, \xi_2)$, as a function of the real control parameters (ξ_1, ξ_2), also has an algebraic branch point of square root type (rank one) at the exceptional point, and also has a branch cut along a line $\mathcal{L}_I$, that extends in the *negative* direction defined by a unit vector $\vec{\xi}_c$ satisfying $(\vec{I} \cdot \vec{\xi}_c) = 0$.

3. The contact equivalent approximant $\hat{\mathcal{E}}_{n,n+1}(\xi_1, \xi_2)$ to the pole position function allowed us to obtain contact equivalent approximants to the resonance energy eigenvalues, $\hat{\mathcal{E}}_n(\xi_1, \xi_2)$ and $\hat{\mathcal{E}}_{n+1}(\xi_1, \xi_2)$, which are zeroes (roots) of a universal unfolding $\hat{f}_{\rm doub}(-k; \xi_1, \xi_2)$ of the Jost function $f(-k; \xi_1, \xi_2)$ at the exceptional point. Therefore we are justified in saying that the family of functions $\hat{\mathcal{E}}_n(\xi_1, \xi_2)$ and $\hat{\mathcal{E}}_{n+1}(\xi_1, \xi_2)$ is a universal unfolding or deformation of any degeneracy or crossing point of two unbound state energy eigenvalues. The family of functions $\hat{\mathcal{E}}_n(\xi_1, \xi_2)$ and $\hat{\mathcal{E}}_{n+1}(\xi_1, \xi_2)$, is contact equivalent to the exact energy eigenvalues, $\mathcal{E}_n(\xi_1, \xi_2)$ and $\mathcal{E}_{n+1}(\xi_1, \xi_2)$, of the isolated doublet of resonances at the exceptional point, and includes all small perturbations of the degeneracy conditions up to contact equivalence
4. The experimentally determined dependence of the difference of resonance energies, $\Delta E = (\text{Re}\, \mathcal{E}_n - \text{Re}\, \mathcal{E}_{n+1})$, and widths, $\Delta\Gamma = (\text{Im}\, \mathcal{E}_{n+1} - \text{Im}\, \mathcal{E}_n)$, on one control parameter, ξ_1, when the other is kept constant, $\xi_2 = \bar{\xi}_2^{(i)}$, may be translated into a geometric language as projections of intersections of the energy hypersurface representing the difference of resonant energy eigenvalues and a hyperplane $\xi_2 = \bar{\xi}_2^{(i)}$, in a Euclidean space $\mathcal{E}_4$ with Cartesian coordinates $(\text{Re}\, \mathcal{E}, \text{Im}\, \mathcal{E}, \xi_1, \xi_2)$.

   The explicit expressions found for the contact equivalent approximants $\hat{\mathcal{E}}_n(\xi_1, \xi_2)$ and $\hat{\mathcal{E}}_{n+1}(\xi_1, \xi_2)$ allowed us to compute the intersections of the hypersurface representing the difference of resonant energy eigenvalues and the hyperplanes, we also compute the projections of the intersections. We found that our geometrically computed and the experimentally determined properties of the differences of resonance energies, ΔE, and widths $\Delta\Gamma$, are in excellent agreement.

In conclusion, the rich phenomenology of crossings and anticrossings of the energies and widths of the resonances of an isolated doublet of unbound states of a quantum system, observed when one control parameter is varied and the other is kept constant, is fully explained in terms of the topology of the energy hypersurface representing the complex resonance energy eigenvalues as functions of the control parameters of the system in the vicinity of the crossing point.

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